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SUPER-RESOLUTION RECONSTRUCTION OF AN IMAGE

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Abstract

This paper presents a generalization of restoration theory for the problem of Super-Resolution Reconstruction (SRR) of an image. In the SRR problem, a set of low quality images is given, and a single improved quality image which fuses their information is required. We present a model for this problem, and show how the classic restoration theory tools - ML, MAP and POCS - can be applied as a solution. A hybrid algorithm which joins the POCS and the ML benefits is suggested.

1. Introduction

In the classic restoration problem in image processing, a blurred and noisy image is given and the purpose is to somehow restore the ideal image prior to the degradation effects. Such problem is typically modeled using the linear vector-matrix equation (using lexicographic ordering for the images [1]):

$$\underline{Y} = H\underline{X} + \underline{N} \ ; \ \underline{N} \sim G\left\{\underline{0}, W^{-1}\right\} \tag{1}$$

The three main tools that have been proposed to solve the above restoration problem are the Maximum Likelihood Estimator (ML) and the Maximum A-postiriori Probability Estimator (MAP), which apply stochastic perception to the problem, and Projection Onto Convex Sets (POCS), which applies set theory tools instead [1].

<u>This paper presents a utilization of the above three estimation tools for the (SRR) problem.</u> In this problem, an improved resolution image is reconstructed based on several geometrically warped, linearly blurred, uniformly down-sampled and noisy measured images. N such measured images are given, and the purpose is to reconstruct a single super-resolution image, which fuses all the measurements into it. The SRR problem have been proposed and treated by several authors in the last decade [2-6]. Among the various proposed methods, the most general and thorough approaches are the Iterative Back Propagation algorithm proposed by Irani & Peleg [4], and the POCS based reconstruction, proposed by Patti, Sezan and Tekalp [5].

This paper is organized as follows: Section 2 presents a new vector-matrix model for the SRR problem, generalizing the above restoration model, shown in equation (1). Using this model, Section 3 presents the ML, MAP and POCS solutions for the SRR problem. In Section 4 a hybrid algorithm which combines the benefits of both the ML and the POCS is presented. Section 5 shows simulation results, and Section 6 concludes this paper.

2. Modeling The Super-Resolution Problem

Given are N measured images $\{\underline{Y}_k\}_{k=1}^{N}$ of different sizes $[M_k \times M_k]$. We assume that these images are different representations of a single high-resolution image $\underline{X}$ of size $[L \times L]$, where typically - $L > M_k$ for $1 \le k \le N$. Each measured image is the result of an arbitrary geometric warping $[L^2 \times L^2]$ matrix F_k, linear space variant blurring $[L^2 \times L^2]$ matrix H_k and uniform rational decimating $[M_k^2 \times L^2]$ matrix D_k performed on the ideal high-resolution image $\underline{X}$. We further assume that each of the measured images is contaminated by zero mean additive Gaussian noise vector $\underline{E}_k$ with auto-correlation $[M_k^2 \times M_k^2]$ matrix W_k^{-1} These noise vectors are uncorrelated between different measurements. Translating the above description to an analytical model we get:

$$\underline{Y}_k = D_k H_k F_k \underline{X} + \underline{E}_k \quad \text{for} \quad 1 \le k \le N \tag{2}$$

All these matrices (F_k, H_k, D_k, W_k) are assumed to be known in advance. Justifying such an assumption is treated in [6]. Having the above model, grouping the N equations into one can be done for notational convenience. This way we get:

$$\begin{bmatrix} \underline{Y}_1 \\ \vdots \\ \underline{Y}_N \end{bmatrix} = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix} \underline{X} + \begin{bmatrix} \underline{E}_1 \\ \vdots \\ \underline{E}_N \end{bmatrix} = \mathbf{H}\underline{X} + \underline{\mathbf{E}} \tag{3}$$

where we have defined $C_k = D_k H_k F_k$, and the auto-correlation of the Gaussian random vector $\underline{\mathbf{E}}$ is:

$$E\{\underline{\mathbf{E}}\underline{\mathbf{E}}^*\} = \begin{bmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_N \end{bmatrix}^{-1} = \mathbf{W}^{-1} \quad (4)$$

The final obtained model equation $\underline{Y} = \mathbf{H}\underline{X} + \underline{\mathbf{E}}$ is a classic restoration problem model [1-2]. Thus, we can easily apply the Maximum Likelihood estimator, the Maximum A-postiriori estimator or the POCS methods in order to restore the image $\underline{X}$, which is exactly our purpose here. In the following sub-sections we shall briefly present the way to specifically apply those tools.

3. Solving the Super-Resolution Problem

Applying the **ML solution** [6] we get:

$$\underline{\hat{X}}_{ML} = \underset{\underline{X}}{\arg\min}\left\{[\underline{Y} - \mathbf{H}\underline{X}]^* \mathbf{W}[\underline{Y} - \mathbf{H}\underline{X}]\right\} \quad (5)$$

which gives the well-known pseudo-inverse result:

$$\mathbf{R}\underline{\hat{X}}_{ML} = \underline{\mathbf{P}}$$
$$\mathbf{R} = \mathbf{H}^*\mathbf{W}\mathbf{H} = \sum_{k=1}^{N} C_k^* W_k C_k \quad (6)$$
$$\underline{\mathbf{P}} = \mathbf{H}^*\mathbf{W}\underline{Y} = \sum_{k=1}^{N} C_k^* W_k \underline{Y}_k$$

Locally adaptive regularization can be included in the above analysis with both algebraic and physical interpretations [1,6]. Using the Laplacian operator S and a weighting matrix V (penalizing non-smoothness according to the a-priori knowledge on the smoothness required at each pixel), we get:

$$\underline{\hat{X}}_{ML} = \underset{\underline{X}}{\arg\min}\left\{[\underline{Y} - \mathbf{H}\underline{X}]^* \mathbf{W}[\underline{Y} - \mathbf{H}\underline{X}] + \beta[S\underline{X}]^* V[S\underline{X}]\right\} \quad (7)$$

Again we get - $\mathbf{R}\underline{\hat{X}}_{ML} = \underline{\mathbf{P}}$, but a new term $\beta S^* V S$ is added to the matrix **R**.

If we assume that the unknown $\underline{X}$ is a zero mean Gaussian random process with auto-correlation matrix Q, the **MAP estimator** becomes the Minimum Mean Square Error estimator. Performing several algebraic steps [6] gives:

$$\underline{\hat{X}}_{MAP} = \underset{\underline{X}}{\arg\min}\left\{[\underline{Y} - \mathbf{H}\underline{X}]^* \mathbf{W}[\underline{Y} - \mathbf{H}\underline{X}] + \underline{X}^* Q^{-1} \underline{X}\right\} \quad (8)$$

Minimizing the above function with respect to $\underline{X}$ yields the following result:

$$\mathbf{R}\underline{\hat{X}}_{MAP} = \underline{\mathbf{P}}$$
$$\mathbf{R} = Q^{-1} + \mathbf{H}^*\mathbf{W}\mathbf{H} = Q^{-1} + \sum_{k=1}^{N} C_k^* W_k C_k \quad (9)$$
$$\underline{\mathbf{P}} = \mathbf{H}^*\mathbf{W}\underline{Y} = \sum_{k=1}^{N} C_k^* W_k \underline{Y}_k$$

and the resemblance to the ML result is evident. It can be shown [1,6] that if an Auto-Regresive model is assumed on the image $\underline{X}$, a simple and direct connection between the Laplacian regularization matrix and the AR coefficients can be established.

The ML, the MAP estimator reduces to a huge sparse set of equations which can be solved iteratively [7].

According to the **set theoretic approach** [1,6], each a-priori knowledge on the required restored image should be formulated as a constraining convex set containing the restored image as a point within this set. Using the model presented earlier, we can suggest a group of such convex sets based on L_2 distance measure:

$$G_k = \left\{\underline{X} \,\middle|\, \left\|D_k H_k F_k \underline{X} - \underline{Y}_k\right\|_{W_k}^2 \le 1\right\} \quad (10)$$

for $1 \le k \le N$. This defines a group of N convex sets - ellipsoids in this case. Since POCS requires a projection onto these sets, and since projection onto an ellipsoid is computationally very complex, L_∞ constraints can be proposed instead [6]:

$$G_k(m,n) = \left\{\underline{X} \,\middle|\, \left|D_k H_k F_k \underline{X}\big|_{(m,n)} - y_k(m,n)\right| \le \delta_k(m,n)\right\} \quad 1 \le k \le N; \quad \forall (m,n) \in \theta_k \quad (11)$$

where θ_k is the support region of the k-th measured image, and δ_k stands for the uncertainty of the model.

Another set which can be used is the one constraining smoothness. We can suggest L_2 or L_∞ convex set versions as before:

$$G_S(m,n) = \left\{ \underline{X} \, \left| \, |S\underline{X}|_{(m,n)} \leq \delta_s \right. \right\} \quad \forall (m,n) \in \theta_0$$
$$\text{or} \quad G_S = \left\{ \underline{X} \, \left| \, \|S\underline{X}\|_V^2 \leq 1 \right. \right\} \tag{12}$$

where θ_0 is the support region of the ideal image. We can incorporate additional non-linear constraints such as constraints on the output energy, phase, support and others. An often used constraint is the one posed on the amplitude of the result:

$$G_A = \left\{ x(m,n) \mid A_1 \leq x(m,n) \leq A_2 \right\} \tag{13}$$

Having a group of M convex sets, each containing the required image, the Projection Onto Convex Set (POCS) method suggests the following iterative algorithm for the recovery of a point within the intersection of these sets [1,6]:

$$\underline{X}_{k+1} = P_M P_{M-1} \cdots P_2 P_1 \left\{ \underline{X}_k \right\} \tag{14}$$

where P_j is the projection of a given point onto the j-th convex set.

A different approach towards the POCS idea is the bounding ellipsoid method [6]. For the case where all the constraints are ellipsoids this approach suggests finding the ellipsoid bounding the intersection of all the participating constraints, and to choosing its center as the output result. In [6] it is shown that the equation for the bounding ellipsoid center is exactly (for a specific case) the ML solution as given in equation (7).

4. The Hybrid Reconstruction Algorithm

While the ML and the MAP are numerically simpler to apply, the POCS is more general and can incorporate non-linear constraints into the reconstruction process as well. In order to gain both these properties, a hybrid algorithm is proposed. We start by defining a new convex optimization problem which combines a quadratic scalar error with M convex constraints:

$$\varepsilon^2 = \left\{ [\underline{Y} - \mathbf{H}\underline{X}]^* \mathbf{W} [\underline{Y} - \mathbf{H}\underline{X}] + \beta [S\underline{X}]^* V [S\underline{X}] \right\}$$
$$\text{subject to} \quad \left\{ \underline{X} \in C_k \quad 1 \leq k \leq M \right\} \tag{15}$$

where the quadratic error takes care of the model and the smoothness errors, and the M additional constraints refer to the non-ellipsoids a-priori knowledge.

Following the iterative methods presented by [1], we propose a simple yet effective two-phase iterative algorithm to solve the above optimization problem. Analysis of this method can be found in [6]. Suppose that an efficient iterative algorithm which is known to converge to the minimum of the scalar squared error is given - denoted by I_t. Algorithms such as the Conjugate Gradient or the Gauss-Siedel can be considered as excellent candidates for I_t. Beyond this first iterative algorithm I_t, M projection operators denoted by J_t^k $k \in [1 \div M]$ can be constructed, each of them projects onto a convex set representing a given constraint. Assuming that the M projections are all given using the Euclidean metric, we suggest the following global iterative step:

$$\underline{X}_{k+1} = J_t^M \left\{ J_t^{M-1} \left\{ \cdots\cdots J_t^1 \left\{ I_t \left\{ \underline{X}_k \right\} \right\} \right\} \right\} \tag{16}$$

This interlaced approach is generally converging to the ***sub-optimal*** point of the problem given in equation (15). Adding several new iterations, where now I_t is replaced by the (notoriously slow [7]) ***Steepest Descent***, updates the previous result and assures that the final convergence is to the optimal point, as is proved in [6].

5. Simulation Results

In this section we present elementary example which demonstrate the effectiveness of the proposed method for the super-resolution restoration problem. A single $[100 \times 100]$ image was taken (gray values in the range 0-63), and from it we have generated 16 blurred, down-sampled and noisy images of size $[50 \times 50]$. The degradation includes random affine motion (with zoom in the range 0.9÷1.1, rotation in the range 0÷50°, and translation in the range -5÷5), blur with the 1-D separable kernel $[0.7 \; 1.0 \; 0.7]/2.4$, a 2:1 decimation ratio, and additive white Gaussian noise with $\sigma = 3$. Figure 1 presents the ideal image, Figure 2 presents 4 images from the measurements, and Figure 3 show the reconstructed image using the hybrid restoration algorithm with regularization. Beyond the ellipsoids forming the quadratic error, the amplitude constraint was applied. 10 GS iterations followed by 30 SD iterations were applied. The regularization approach is explained in details in [6].

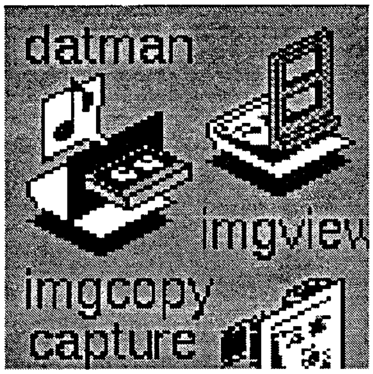

Figure 1 - The ideal image

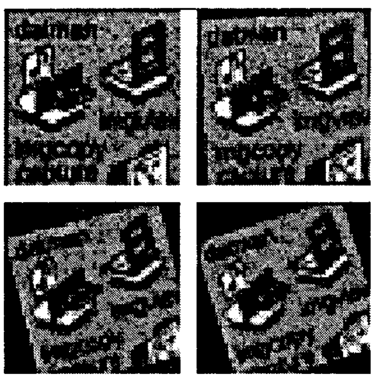

Figure 2 - Four images from the measurements

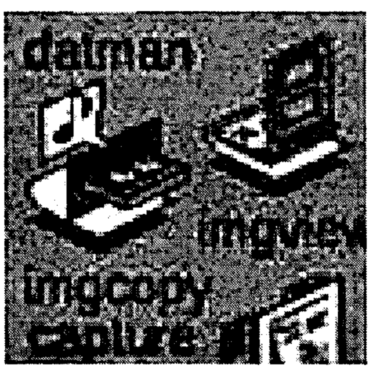

Figure 3 - A reconstructed result

6. Conclusion

This paper presents the problem of Super-Resolution Reconstruction, and its solution. A new general model is introduced, which enables a direct generalization of the classic tools from restoration theory - the ML, the MAP and the POCS methods. A hybrid algorithm is proposed which combines the benefits of the simple ML estimator, and the ability of the POCS to incorporate non-ellipsoids constraints. An efficient iterative two phases algorithm is presented to solve the new defined problem, and convergence is assured to the optimal point.

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