Sparse Representations of Signals: Theory and Applications *

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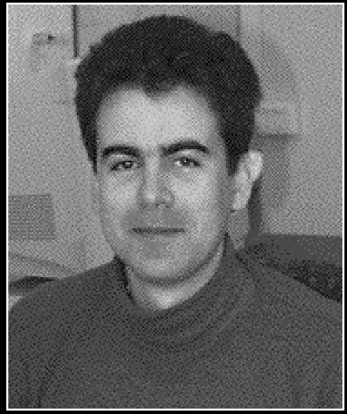

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Agenda

1. **Introduction**
   Sparse & overcomplete representations, pursuit algorithms

2. Success of BP/MP as Forward Transforms
   Uniqueness, equivalence of BP and MP

3. Success of BP/MP for Inverse Problems
   Uniqueness, stability of BP and MP

4. Applications
   Image separation and inpainting

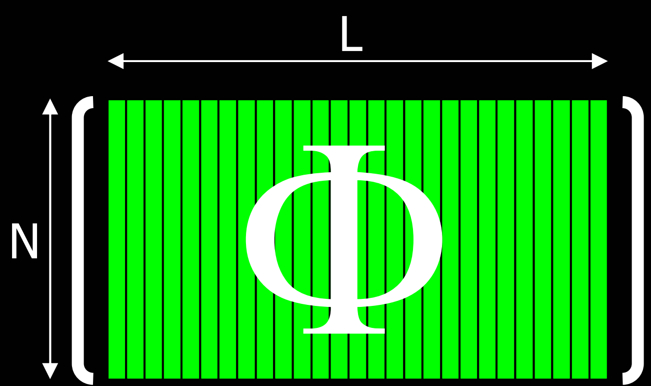
Problem Setting – Linear Algebra

Our dream – solve an linear system of equations of the form

$$\underline{x} = \Phi\underline{\alpha}$$

known ($\underline{x}$ and Φ)

where



- L>N,
- Φ is full rank, and
- Columns are normalized

Can We Solve This?

Generally NO* !

* Unless additional information is introduced.

Our assumption for today:

the sparsest possible solution is preferred

Great … But,

- Why look at this problem at all? What is it good for? Why sparseness?
- Is now the problem well defined now? does it lead to a unique solution?
- How shall we numerically solve this problem?

These and related questions will be discussed in today's talk

Addressing the First Question

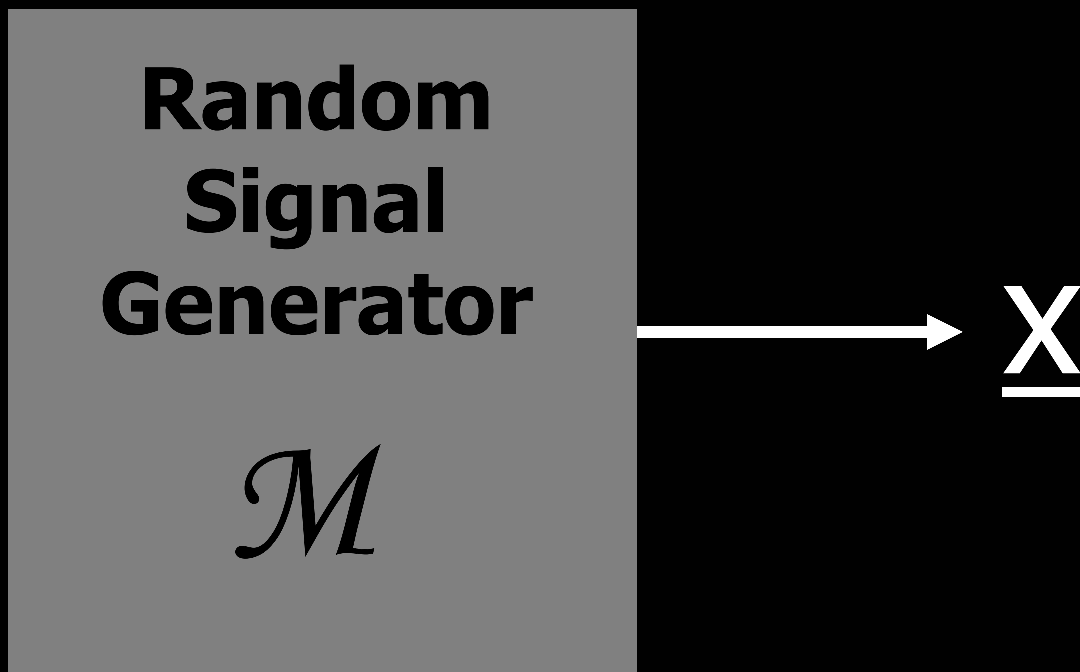
We will use the linear relation

$$\underline{x} = \Phi \underline{\alpha}$$

as the core idea for modeling signals

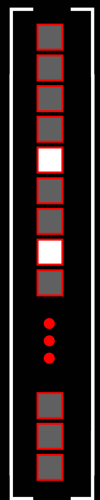
Signals' Origin in **Sparse-Land**

We shall assume that our signals of interest emerge from a random generator machine $\mathcal{M}$



Signals' Origin in Sparse-Land

$\underline{\alpha} =$
sparse

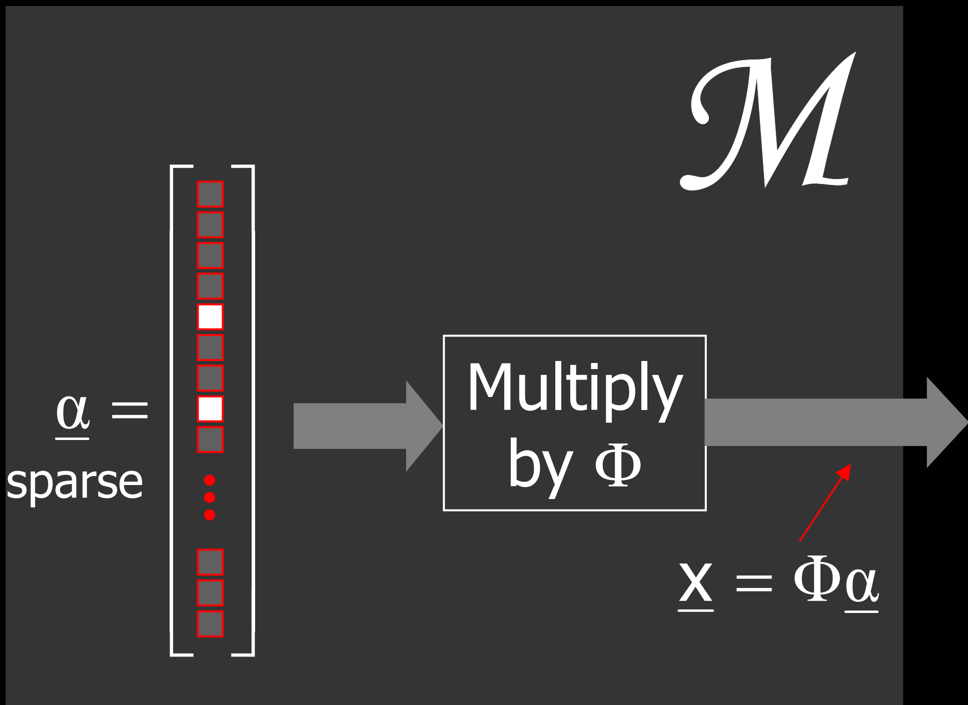


Instead of defining $\mathcal{M}$ over the signals directly, we define it over "their representations" $\underline{\alpha}$:

- Draw the number of none-zeros (s) in $\underline{\alpha}$ with probability P(s),
- Draw the s locations from L independently,
- Draw the weights in these s locations independently (Gaussian/Laplacian).

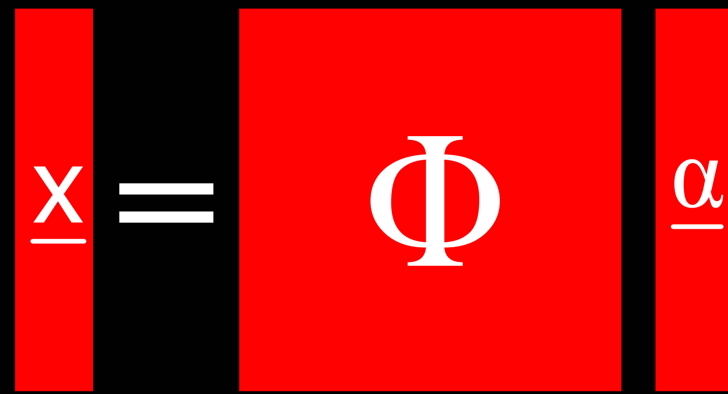
The obtained vectors are very simple to generate or describe.

Signals' Origin in Sparse-Land



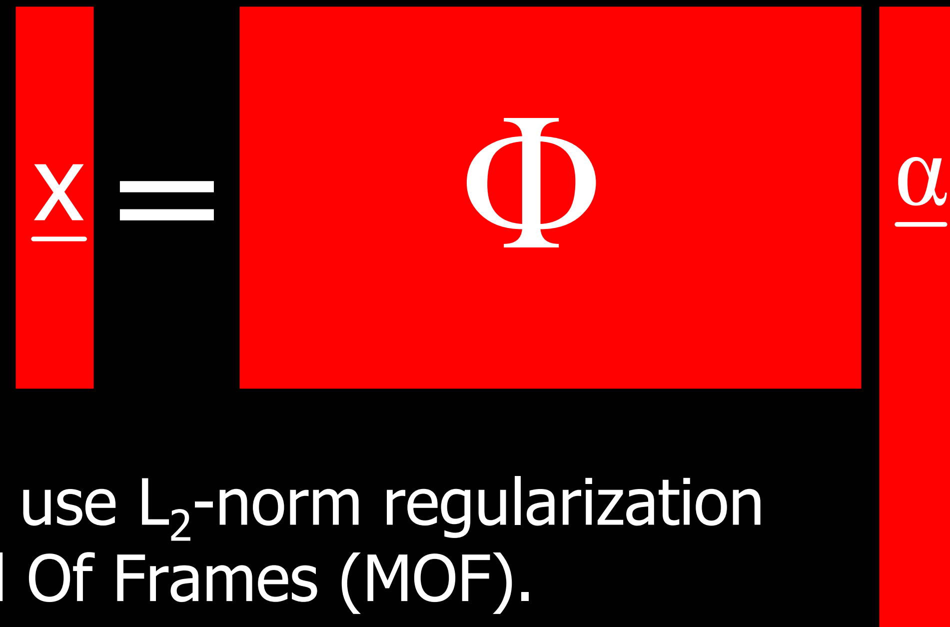
- Every generated signal is built as a linear combination of few columns (atoms) from our dictionary Φ
- The obtained signals are a special type mixture-of-Gaussians (or Laplacians) – every column participate as a principle direction in the construction of many Gaussians

Why This Model?



- For a square system with non-singular Φ, there is no need for sparsity assumption.
- Such systems are commonly used (DFT, DCT, wavelet, …).
- Still, we are taught to prefer 'sparse' representations over such systems (N-term approximation, …).
- We often use signal models defined via the transform coefficients, assumed to have a simple structure (e.g., independence).

Why This Model?



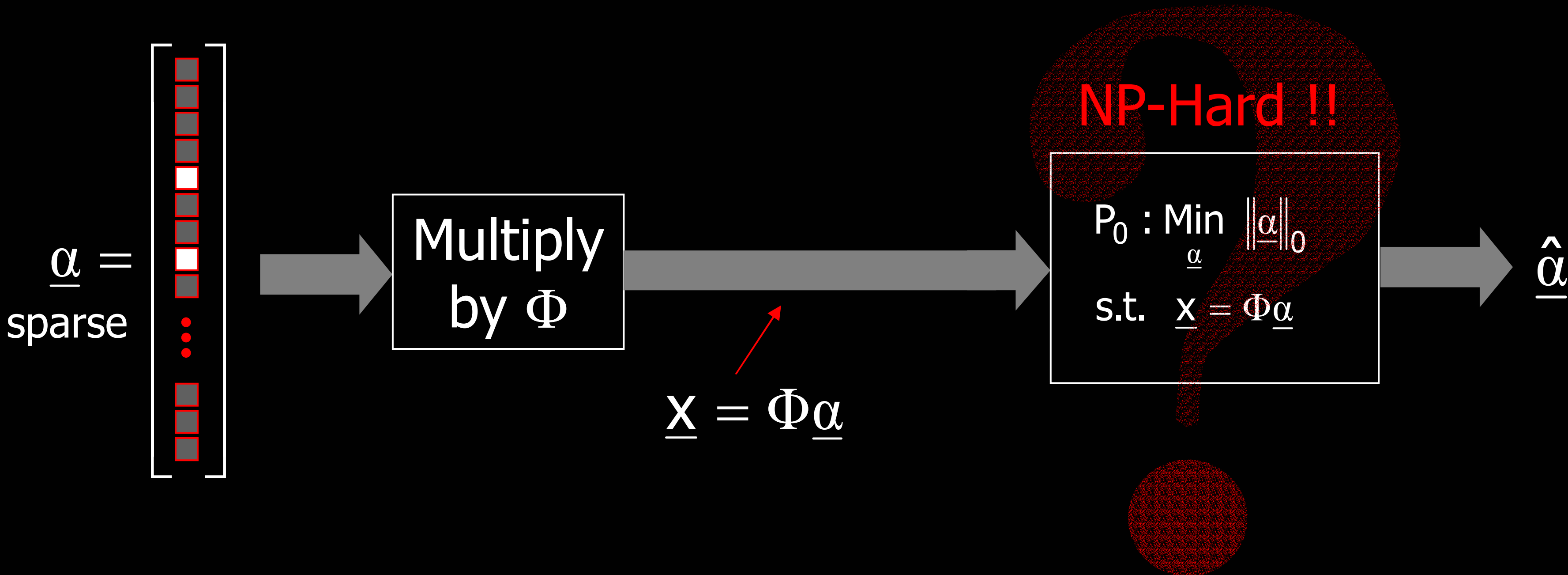
- Going over-complete has been also considered in past work, in an attempt to strengthen the sparseness potential.
- Such approaches generally use L_2-norm regularization to go from $\underline{x}$ to $\underline{\alpha}$ – Method Of Frames (MOF).
- Bottom line: The model presented here is in line with these attempts, trying to address the desire for sparsity directly, while assuming independent coefficients in the 'transform domain'.

What's to do With Such a Model?

- **Signal Transform:** Given the signal, its sparsest (over-complete) representation $\underline{\alpha}$ is its forward transform. Consider this for compression, feature extraction, analysis/synthesis of signals, …

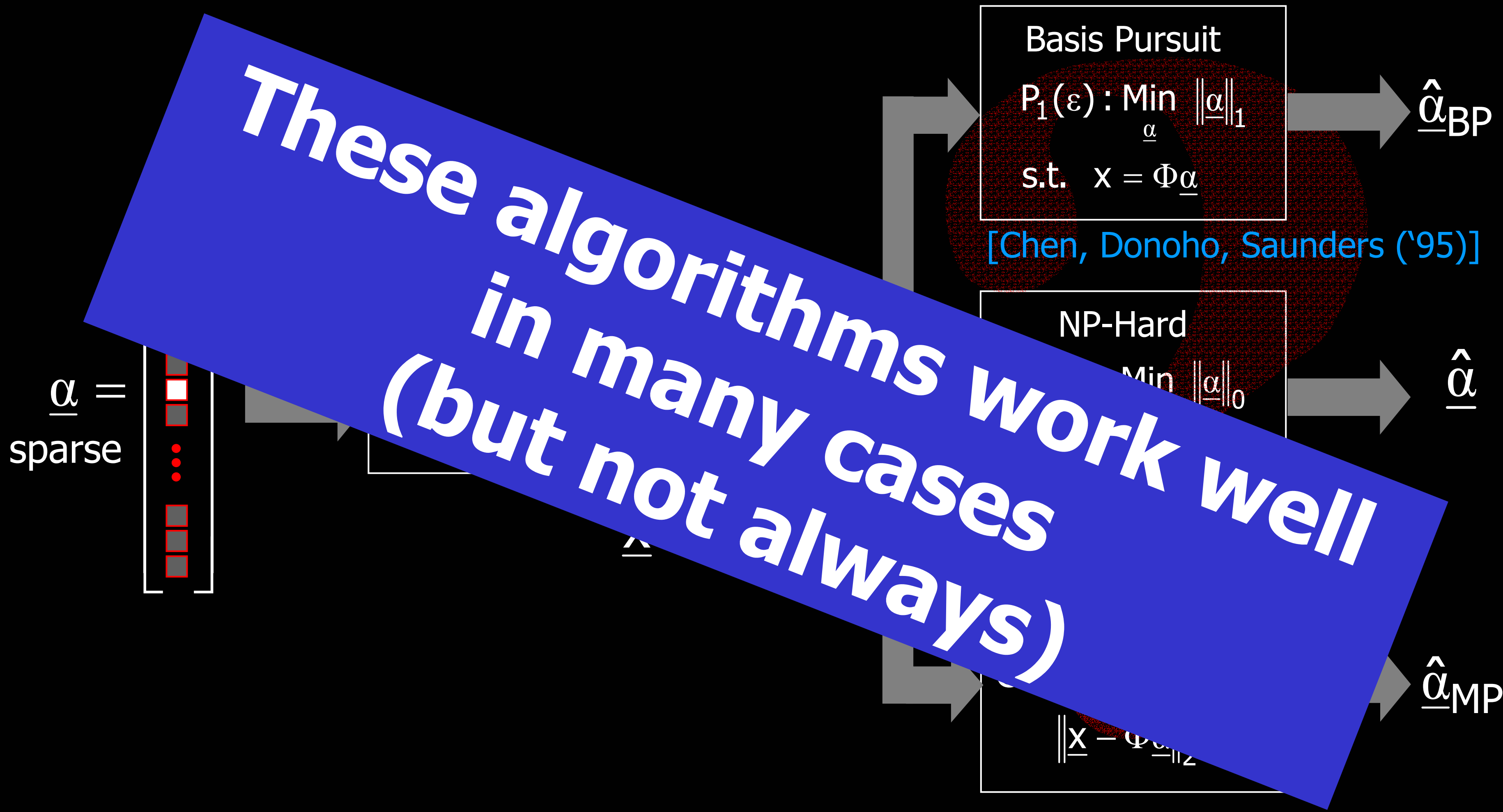
- **Signal Prior:** in inverse problems seek a solution that has a sparse representation over a predetermined dictionary, and this way regularize the problem (just as TV, bilateral, Beltrami flow, wavelet, and other priors are used).

Signal's Transform



- Is $\hat{\underline{\alpha}} = \underline{\alpha}$? Under which conditions?
- Are there practical ways to get $\hat{\underline{\alpha}}$?
- How effective are those ways?

Practical Pursuit Algorithms



These algorithms work well in many cases (but not always)

Signal Prior

- Assume that $\underline{x}$ is known to emerge from $\mathcal{M}$, i.e. $\exists\underline{\alpha}$ sparse such that

$$\underline{x} = \Phi\underline{\alpha}$$

- Suppose we observe $\underline{y} = \underline{x} + \underline{v}$, a noisy version of $\underline{x}$ with $\|\underline{v}\|_2 \leq \varepsilon$.
- We denoise the signal $\underline{y}$ by solving

$$P_0(\varepsilon): \quad \underset{\underline{\alpha}}{\text{Min}} \|\underline{\alpha}\|_0 \quad \text{s.t.} \quad \|\underline{y} - \Phi\underline{\alpha}\|_2 \leq \varepsilon$$

- This way we see that sparse representations can serve in inverse problems (denoising is the simplest example).

To summarize …

- Given a dictionary Φ and a signal $\underline{x}$, we want to find the sparsest "atom decomposition" of the signal by either

$$\underset{\underline{\alpha}}{\text{Min}} \left\|\underline{\alpha}\right\|_0 \quad \text{s.t.} \quad \underline{x} = \Phi\underline{\alpha} \qquad \text{or} \qquad \underset{\underline{\alpha}}{\text{Min}} \left\|\underline{\alpha}\right\|_0 \quad \text{s.t.} \quad \left\|\underline{x} - \Phi\underline{\alpha}\right\|_2 \leq \varepsilon$$

- Basis/Matching Pursuit algorithms propose alternative traceable method to compute the desired solution.

- Our focus today:
  - Why should this work?
  - Under what conditions could we claim success of BP/MP?
  - What can we do with such results?

Due to the Time Limit …

(and the speaker's limited knowledge) we will NOT discuss today

- Proofs (and there are beautiful and painful  proofs).
- Numerical considerations in the pursuit algorithms.
- Exotic results (e.g. ℓ^p-norm results, amalgam of ortho-bases, uncertainty principles).
- Average performance (probabilistic) bounds.
- How to train on data to obtain the best dictionary Φ.
- Relation to other fields (Machine Learning, ICA, …).

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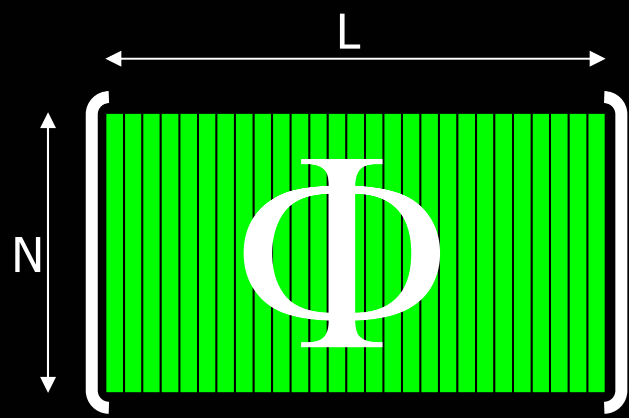
Problem Setting

The Dictionary:



Every column is normalized to have an l_2 unit norm

Our dream - Solve:

$$P_0 : \underset{\underline{\alpha}}{\text{Min}} \left\|\underline{\alpha}\right\|_0 \quad \text{s.t.} \quad \underline{x} = \Phi\underline{\alpha}$$

known

Uniqueness – Matrix "Spark"

Definition*: Given a matrix Φ, σ=Spark{Φ} is the smallest number of columns from Φ that are linearly dependent.

Properties

- Generally: $2 \le \sigma = \text{Spark}\{\Phi\} \le \text{Rank}\{\Phi\}+1$.
- By definition, if $\Phi\underline{v}=0$ then $\|\underline{v}\|_0 \ge \sigma$.
- For any pair of representations of $\underline{x}$ we have

$$\underline{x} = \Phi\underline{\gamma}_1 = \Phi\underline{\gamma}_2 \quad \Rightarrow \quad \Phi\left(\underline{\gamma}_1 - \underline{\gamma}_2\right) = \underline{0} \quad \Rightarrow \quad \left\|\underline{\gamma}_1 - \underline{\gamma}_2\right\|_0 \ge \sigma$$

* Kruskal rank (1977) is defined the same – used for decomposition of tensors (extension of the SVD).

Uniqueness Rule – 1

$$\sigma \le \left\|\underline{\gamma}_1\right\|_0 + \left\|\underline{\gamma}_2\right\|_0$$

Uncertainty rule: Any two different representations of the same $\underline{x}$ cannot be jointly too sparse – the bound depends on the properties of the dictionary.

Result 1

If we found a representation that satisfy

$$\frac{\sigma}{2} > \left\|\underline{\gamma}\right\|_0$$

Then necessarily it is unique (the sparsest).

Donoho & E ('02)

Gribonval & Nielsen ('03)

Malioutov et.al. ('04)

Surprising result! In general optimization tasks, the best we can do is detect and guarantee local minimum.

Evaluating the "Spark"

- Define the "Mutual Incoherence" as

$$\sqrt{\frac{L-N}{N(L+1)}} \le \max_{1\le k,j\le L,k\ne j}\left\{\left|\underline{\varphi}_k^H \underline{\varphi}_j\right|\right\} \le 1$$

- We can show (based on Gerśgorin disks theorem) that a lower-bound on the spark is obtained by

$$\sigma \ge 1+\frac{1}{M}.$$

- Non-tight lower bound – too pessimistic! (Example, for $[I,F_N]$ the lower bound is $1+\sqrt{N}$ instead of $2\sqrt{N}$).

Lower bound obtained by Thomas Strohmer (2003).

Uniqueness Rule – 2

$$1+\frac{1}{M} \le \sigma \le \left\|\underline{\gamma}_1\right\|_0 + \left\|\underline{\gamma}_2\right\|_0$$

This is a direct extension of the previous uncertainly result with the Spark, and the use of the bound on it.

Result 2

If we found a representation that satisfy

$$\frac{\sigma}{2} \ge \frac{1}{2}\left(1+\frac{1}{M}\right) > \left\|\underline{\gamma}\right\|_0$$

Then necessarily it is unique (the sparsest).

Donoho & E ('02)

Gribonval & Nielsen ('03)

Malioutov et.al. ('04)

Uniqueness Implication

- We are interested in solving

$$P_0 : \quad \underset{\underline{\alpha}}{\text{Min}} \; \|\underline{\alpha}\|_0 \;\; \text{s.t.} \;\; \underline{x} = \Phi\underline{\alpha}.$$

- Somehow we obtain a candidate solution $\hat{\underline{\alpha}}$.
- The uniqueness theorem tells us that a simple test on $\hat{\underline{\alpha}}$ could tell us if it is the solution of P_0.

- However:
  - If the test is negative, it says nothing.
  - This does not help in solving P_0.
  - This does not explain why BP/MP may be a good replacements.

BP Equivalence

In order for BP to succeed, we have to show that sparse enough solutions are the smallest also in ℓ^1-norm. Using duality in linear programming one can show the following:

Result 4

Given a signal $\underline{x}$ with a representation $\underline{x} = \Phi\underline{\gamma}$,
Assuming that $\|\underline{\gamma}\|_0 < 0.5(1 + 1/M)$, P_1 (BP) is
Guaranteed to find the sparsest solution*.

Donoho & E ('02)
Gribonval & Nielsen ('03)
Malioutov et.al. ('04)

* Is it a tight result? What is the role of "Spark" in dictating Equivalence?

MP Equivalence

As it turns out, the analysis of the MP is even simpler ! After the results on the BP were presented, both Tropp and Temlyakov shown the following:

Result 5

Tropp ('03)

Temlyakov ('03)

Given a signal $\underline{x}$ with a representation $\underline{x} = \Phi\underline{\gamma}$,

Assuming that $\|\underline{\gamma}\|_0 < 0.5(1 + 1/M)$, MP is

Guaranteed to find the sparsest solution.

SAME RESULTS !?

Are these algorithms really comparable?

To Summarize so far …

Transforming signals from Sparse-Land can be done by seeking their original representation

forward transform?

Use pursuit Algorithms

Why works so well?

We explain (uniqueness and equivalence) – give bounds on performance

Implications?

(a) Design of dictionaries via (M,σ),

(b) Test of solution for optimality,

(c) Use in applications as a forward transform.

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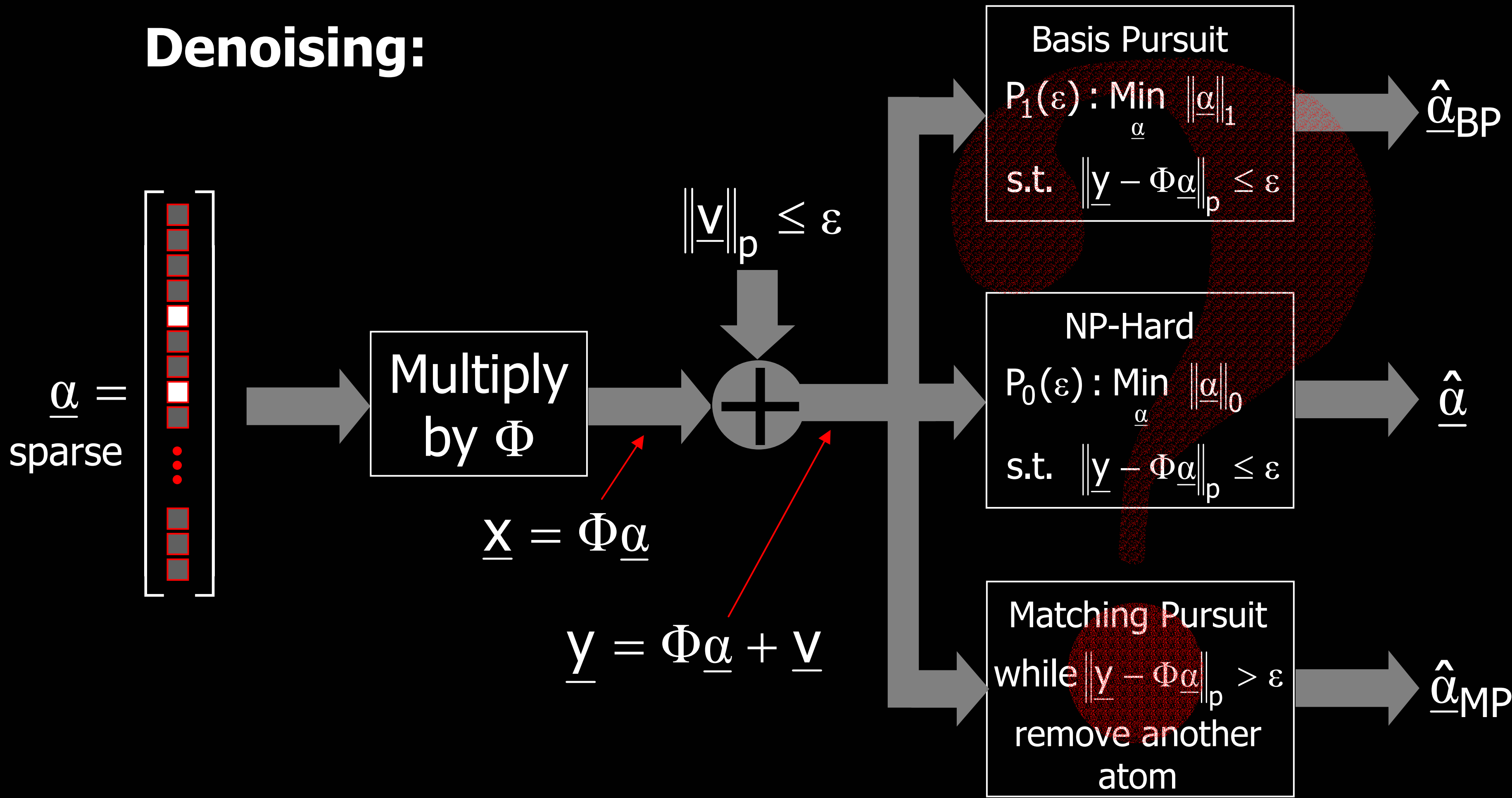
The Simplest Inverse Problem

Denoising:



Questions We Should Ask

- Reconstruction of the signal:
  - What is the relation between this and other Bayesian alternative methods [e.g. TV, wavelet denoising, …]?
  - What is the role of over-completeness and sparsity here?
  - How about other, more general inverse problems?

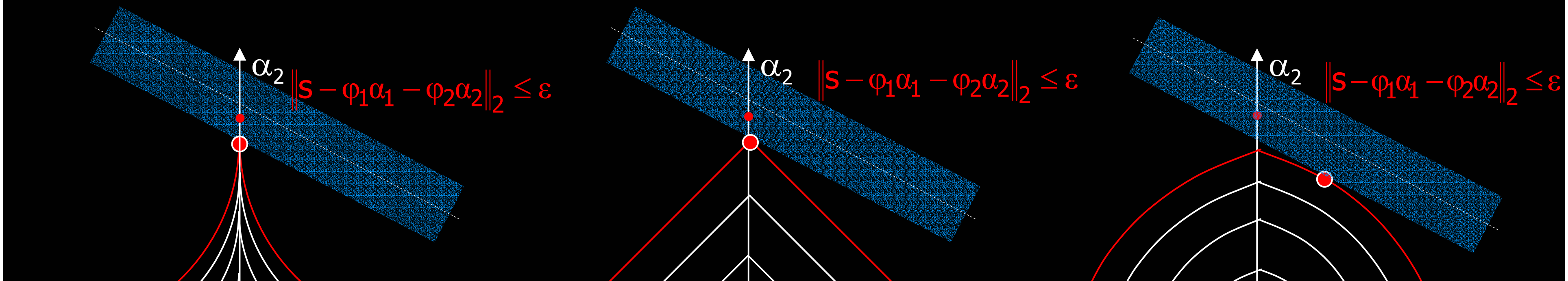
  These are topics of our current research with P. Milanfar, D.L. Donoho, and R. Rubinstein.

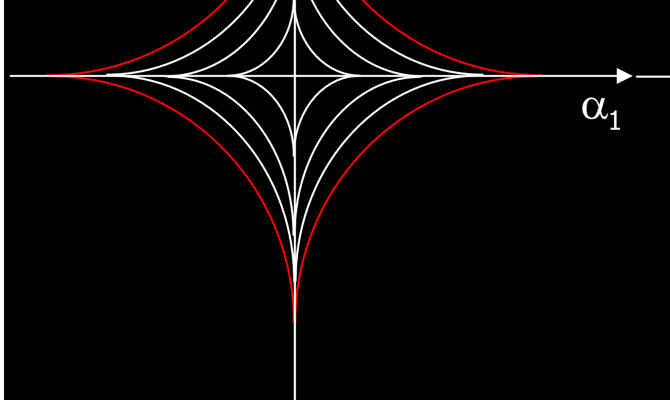
- Reconstruction of the representation:
  - Why the denoising works with $P_0(\varepsilon)$?
  - Why should the pursuit algorithms succeed?

  These questions are generalizations of the previous treatment.

2D–Example

$$\underset{[\alpha_1,\alpha_2]}{\text{Min}} \ |\alpha_1|^p + |\alpha_2|^p \quad \text{s.t.} \quad \|y - \varphi_1\alpha_1 - \varphi_2\alpha_2\|_2 \le \varepsilon$$





$0 \le P < 1$

Intuition Gained:

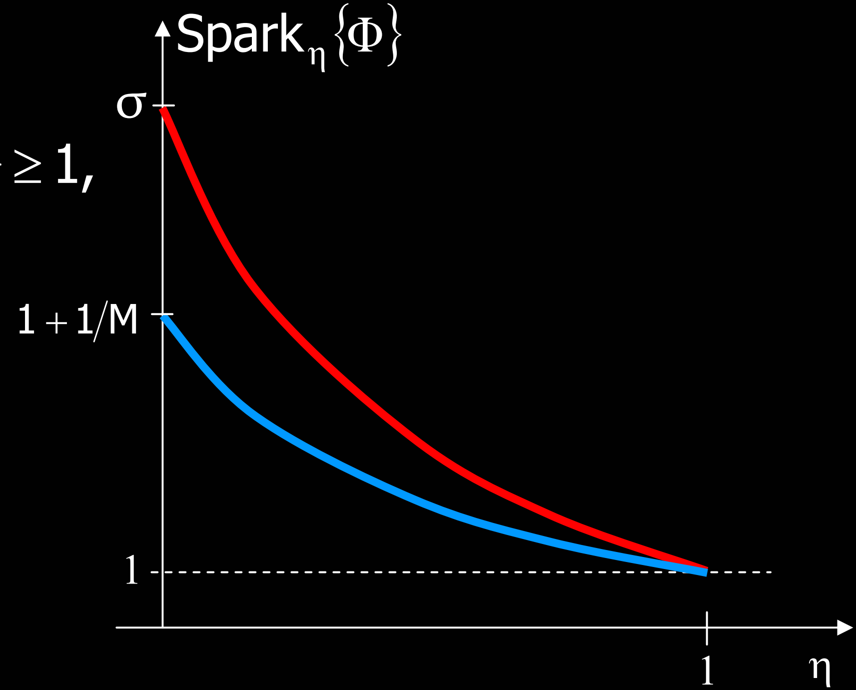
- Exact recovery is unlikely even for an exhaustive P_0 solution.
- Sparse $\underline{\alpha}$ can be recovered well both in terms of support and proximity for $p \le 1$.

Uniqueness? Generalizing Spark

Definition: $\text{Spark}_\eta\{\Phi\}$ is the smallest number of columns from Φ that give a smallest singular value $\leq \eta$.

Properties:

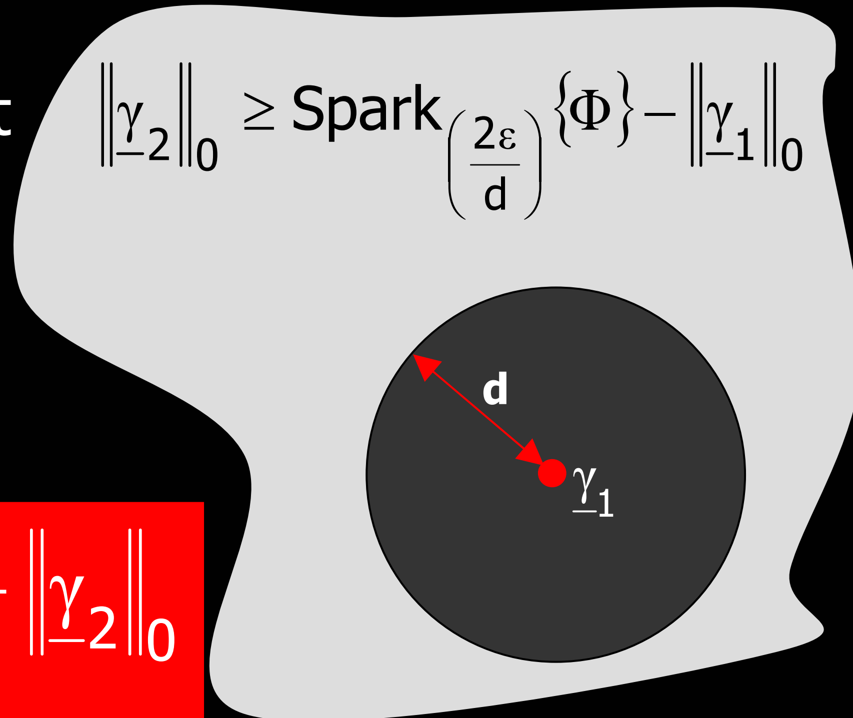
1. For $\eta \geq 0$, $\sigma = \text{Spark}_0\{\Phi\} \geq \text{Spark}_\eta\{\Phi\} \geq 1$,
2. $\text{Spark}_\eta\{\Phi\}$ mon. non-increasing,
3. $\text{Spark}_\eta\{\Phi\} \geq 1 + \left(1 - \eta^2\right)/M$,
4. $\|A\underline{v}\|_2 \leq \eta$ & $\|\underline{v}\|_2 = 1$
   $\Rightarrow \|\underline{v}\|_0 \geq \text{Spark}_\eta\{A\}$.



Generalized Uncertainty Rule

Assume two feasible & different representations of $\underline{y}$:

$$\left\|\underline{y}-\Phi\underline{\gamma}_1\right\|_2 \le \varepsilon \ \ \& \ \ \left\|\underline{y}-\Phi\underline{\gamma}_2\right\|_2 \le \varepsilon$$



Result 6

$$\text{Spark}_{\eta}\{\Phi\} \le \left\|\underline{\gamma}_1\right\|_0 + \left\|\underline{\gamma}_2\right\|_0$$

for $\eta = \dfrac{2\varepsilon}{\left\|\underline{\gamma}_1-\underline{\gamma}_2\right\|_2}$

Donoho, E, & Temlyakov ('04)

The further the candidate alternative from $\underline{\gamma}_1$, the denser is must be.

Uniqueness Rule

Result 7

If we found a representation that satisfy

$$\|\underline{\gamma}\|_0 < \frac{1}{2}\text{Spark}_\eta\{\Phi\}$$

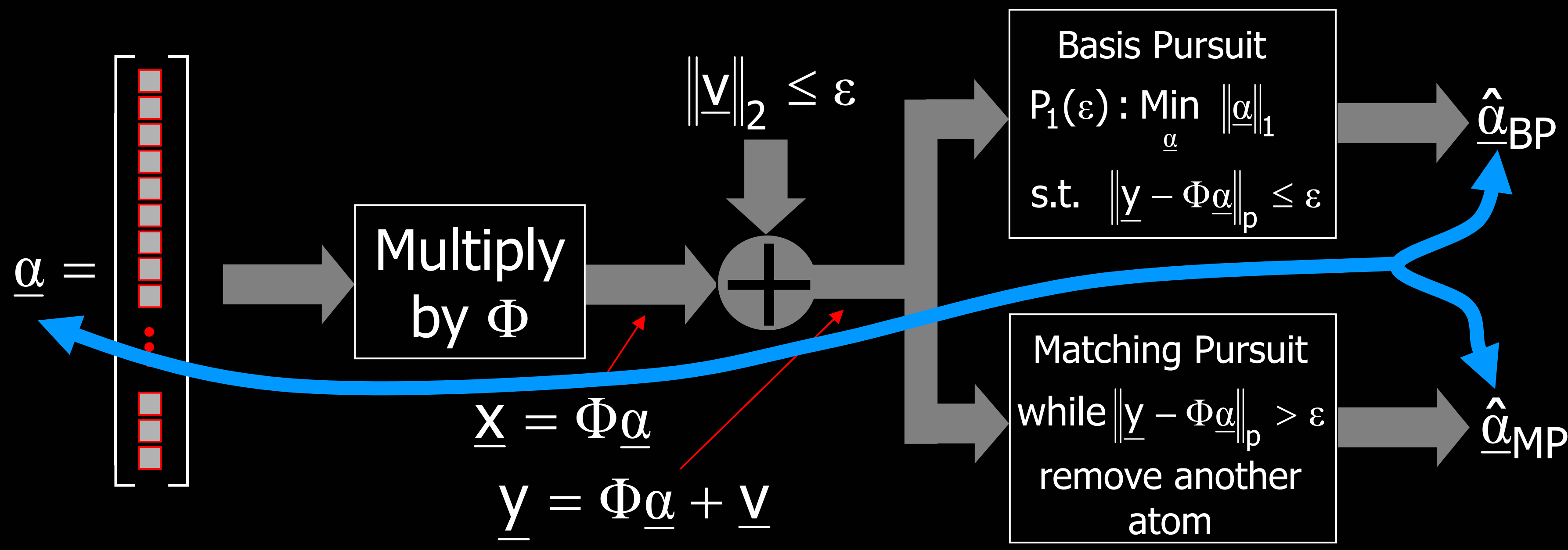
then necessarily it is unique (the sparsest) among all representations that are AT LEAST $2\varepsilon/\eta$ away (in ℓ^2 sense) .

Donoho, E, & Temlyakov ('04)

Implications: 1. This result becomes stronger if we are willing to consider substantially different representations.

2. Put differently, if you found two very sparse approximate representations of the same signal, they must be close to each other.

Are the Pursuit Algorithms Stable?



Stability:
Under which conditions on the original representations $\underline{\alpha}$, could we guarantee that $\|\hat{\underline{\alpha}}_{BP} - \underline{\alpha}\|_2$ and $\|\hat{\underline{\alpha}}_{MP} - \underline{\alpha}\|_2$ are small?

BP Stability

Result 8

Given a signal $\underline{y} = \Phi\underline{\alpha} + \underline{v}$ with a representation satisfying $\|\underline{\alpha}\|_0 < 0.25(1 + 1/M)$ and bounded noise $\|\underline{v}\|_2 \le \varepsilon$, BP will give stability, i.e.,

$$\left\|\underline{\hat{\alpha}}_{BP} - \underline{\alpha}\right\|_2^2 < \frac{4\varepsilon^2}{1 - M(4\|\underline{\alpha}\|_0 + 1)}$$

Donoho, E, & Temlyakov ('04), Tropp ('04), Donoho & E ('04)

Observations: 1. ε=0 – weaker version of previous result

2. Surprising - the error is independent of the SNR, and

3. The result is useless for assessing denoising performance.

MP Stability

Result 9

Given a signal $\underline{y} = \Phi\underline{\alpha} + \underline{v}$ with bounded noise $\|\underline{v}\|_2 \le \varepsilon$, and a sparse representation,

$$\|\underline{\alpha}\|_0 < \frac{1}{2}\left(1 + \frac{1}{M}\right) - \frac{1}{M} \cdot \frac{\varepsilon}{\min_k \left\{|\alpha(k)|\right\}}$$

MP will give stability, i.e.,

$$\|\hat{\underline{\alpha}}_{MP} - \underline{\alpha}\|_2^2 < \frac{\varepsilon^2}{1 - M(\|\underline{\alpha}\|_0 + 1)}$$

Donoho, E, & Temlyakov ('04), Tropp ('04)

Observations: 1. ε=0 leads to the results shown already,

2. Here the error is dependent of the SNR, and

3. There are additional results on the sparsity pattern.

To Summarize This Part …

BP/MP can serve for forward transform of Sparse-Land signals

What about noise?

Relax the equality constraint

Is it still theoretically sound?

We show uncertainty, uniqueness and stability results for the noisy setting

Where next?

- Denoising performance?
- Relation to other methods?
- More general inverse problems?
- Role of over-completeness?
- Average study? Candes & Romberg HW

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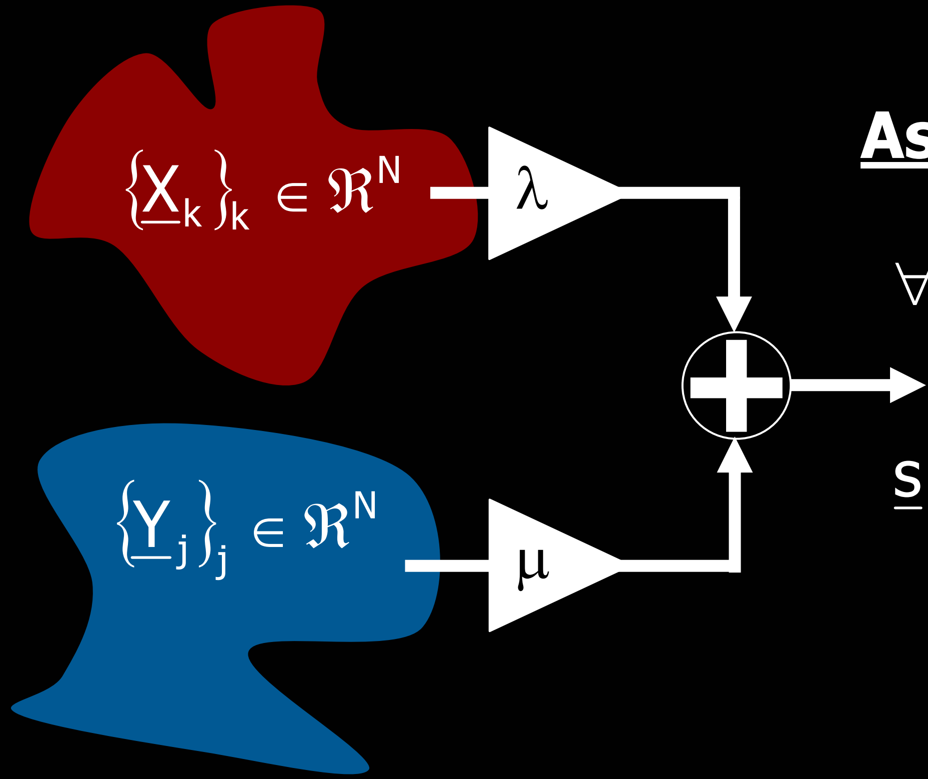
Decomposition of Images

Family of Cartoon images



Family of Texture images

Our
<u>Assumption</u>

$\forall \underline{s} \quad \exists k, j, \lambda, \mu$

such that

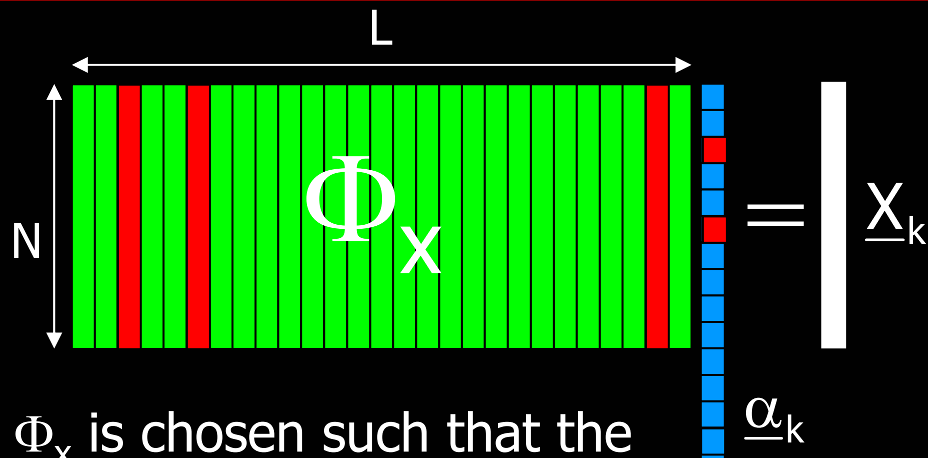
$\underline{s} = \lambda \underline{X}_k + \mu \underline{Y}_j$

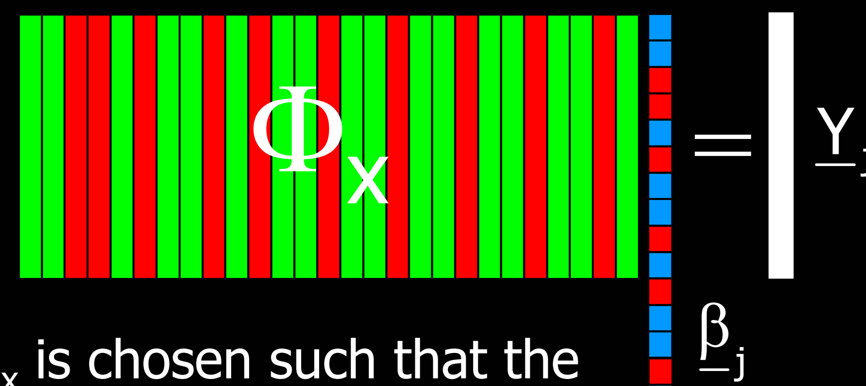
Our Inverse
<u>Problem</u>

Given $\underline{s}$, find its building parts and the mixture weights

$\lambda, \mu, \underline{X}_k, \underline{Y}_j$

Use of Sparsity





Φ_x is chosen such that the representation of $\{\underline{X}_k\}_k \in \Re^N$ are sparse:

$$\left\{ \underline{\alpha}_k = \underset{\underline{\alpha}}{\text{ArgMin}} \|\underline{\alpha}\|_0 \ \text{ s.t. } \ \underline{X}_k = \Phi_x \underline{\alpha} \right\}_k$$

$$\Rightarrow \ \forall k \ \|\underline{\alpha}_k\|_0 << N$$

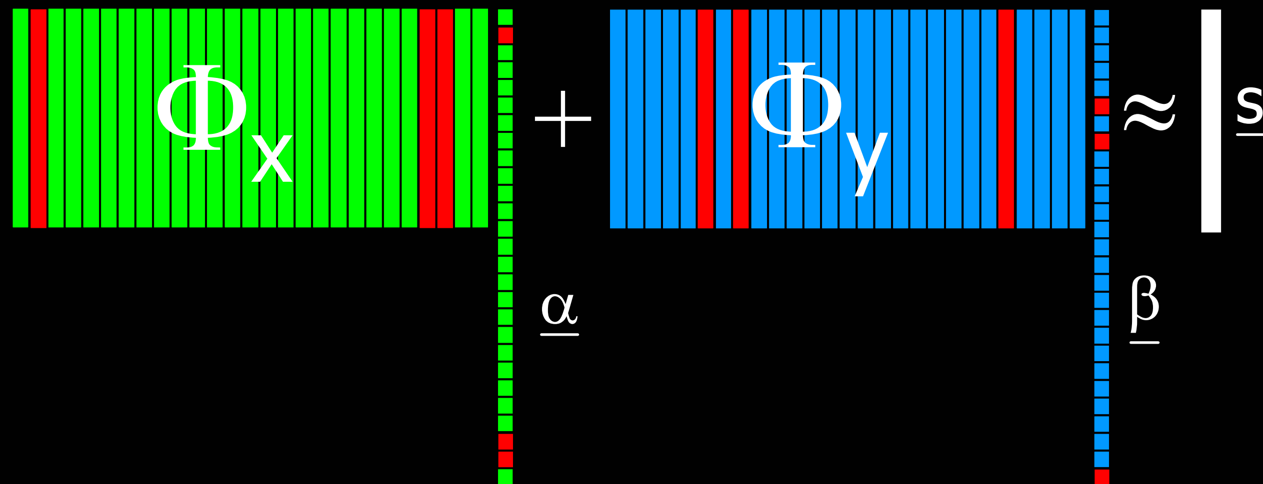
Φ_x is chosen such that the representation of $\{\underline{Y}_j\}_j \in \Re^N$ are non-sparse:

$$\left\{ \underline{\beta}_j = \underset{\underline{\beta}}{\text{ArgMin}} \|\underline{\beta}\|_0 \ \text{ s.t. } \ \underline{Y}_j = \Phi_x \underline{\beta} \right\}_k$$

$$\Rightarrow \ \forall j \ \|\underline{\beta}_j\|_0 \rightarrow N$$

We similarly construct Φ_y to sparsify Y's while being inefficient in representing the X's.

Decomposition via Sparsity



$$\begin{bmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \end{bmatrix} = \underset{\underline{\alpha},\underline{\beta}}{\text{ArgMin}} \ \|\underline{\alpha}\|_1 + \|\underline{\beta}\|_1 \quad \text{s.t.} \quad \left\| \underline{s} - \begin{bmatrix} \Phi_x & \Phi_y \end{bmatrix} \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix} \right\|_2 \le \varepsilon$$

- The idea – if there is a sparse solution, it stands for the separation.
- This formulation removes noise as a by product of the separation.

Theoretical Justification

Several layers of study:

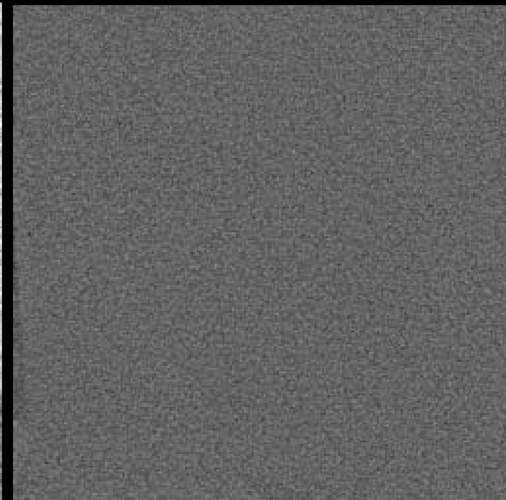
1. Uniqueness/stability as shown above apply directly but are ineffective in handling the realistic scenario where there are many non-zero coefficients.
2. Average performance analysis (Candes & Romberg HW) could remove this shortcoming.
3. Our numerical implementation is done on the “analysis domain” – Donoho’s results apply here.
4. All is built on a **model** for images as being built as sparse combination of $\Phi_x\underline{\alpha}+\Phi_y\underline{\beta}$.

Prior Art

- **Coifman's dream** – The concept of combining transforms to represent efficiently different signal contents was advocated by R. Coifman already in the early 90's.
- **Compression** – Compression algorithms were proposed by F. Meyer et. al. (2002) and Wakin et. al. (2002), based on separate transforms for cartoon and texture.
- **Variational Attempts** – Modeling texture and cartoon and variational-based separation algorithms: E. Meyer (2002), Vese & Osher (2003), Aujol et. al. (2003,2004).
- **Sketchability** – a recent work by Guo, Zhu, and Wu (2003) – MP and MRF modeling for sketch images.

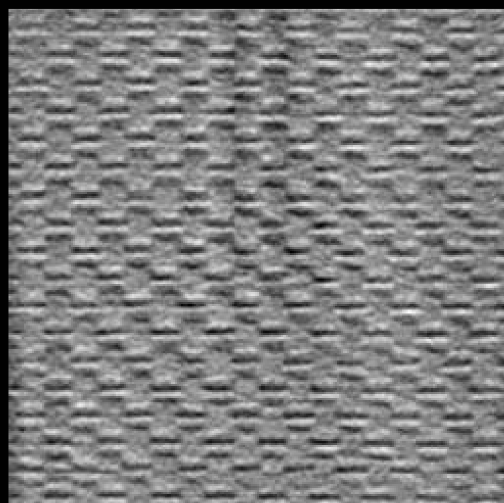
Results – Synthetic + Noise

Original image composed as a combination of texture, cartoon, and additive noise (Gaussian, $\sigma = 10$)



The residual, being the identified noise

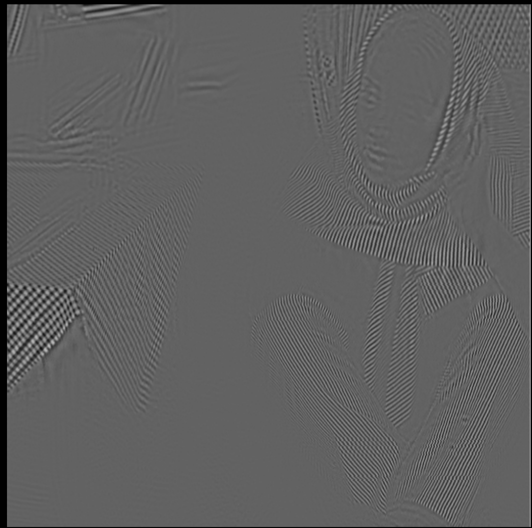
The separated texture (spanned by Global DCT functions)



The separated cartoon (spanned by 5 layer Curvelets functions+LPF)

Results on 'Barbara'

Original 'Barbara' image

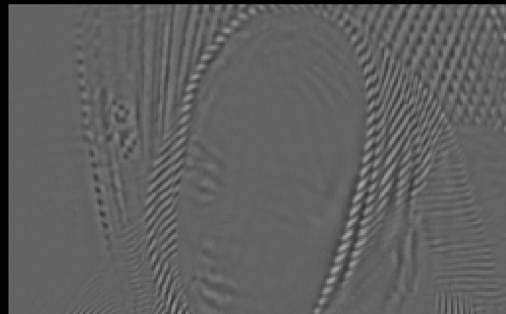


Separated texture using local overlapped DCT (32×32 blocks)

Separated Cartoon using Curvelets (5 resolution layers)

Results – ‘Barbara’ Zoomed in

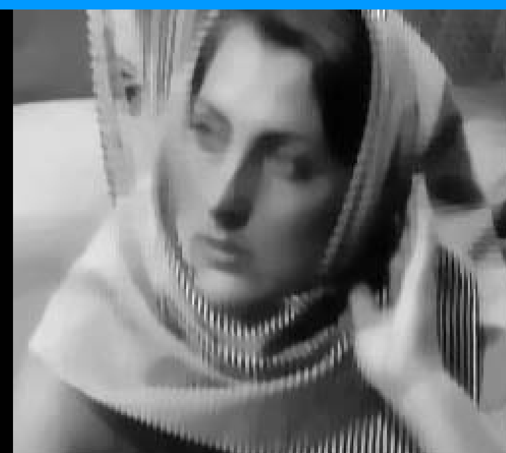
Zoom in on the result shown in the previous slide (the texture part)



The same part taken from Vese’s et. al.

We should note that Vese-Osher algorithm is much faster because of our use of curvelet

Zoom in on the results shown in the previous slide (the cartoon part)



The same part taken from Vese’s et. al.

Inpainting

For separation $\begin{bmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \end{bmatrix} = \underset{\underline{\alpha},\underline{\beta}}{\text{ArgMin}} \|\underline{\alpha}\|_1 + \|\underline{\beta}\|_1 + \lambda \|\underline{s} - \Phi_x \underline{\alpha} - \Phi_y \underline{\beta}\|_2^2$

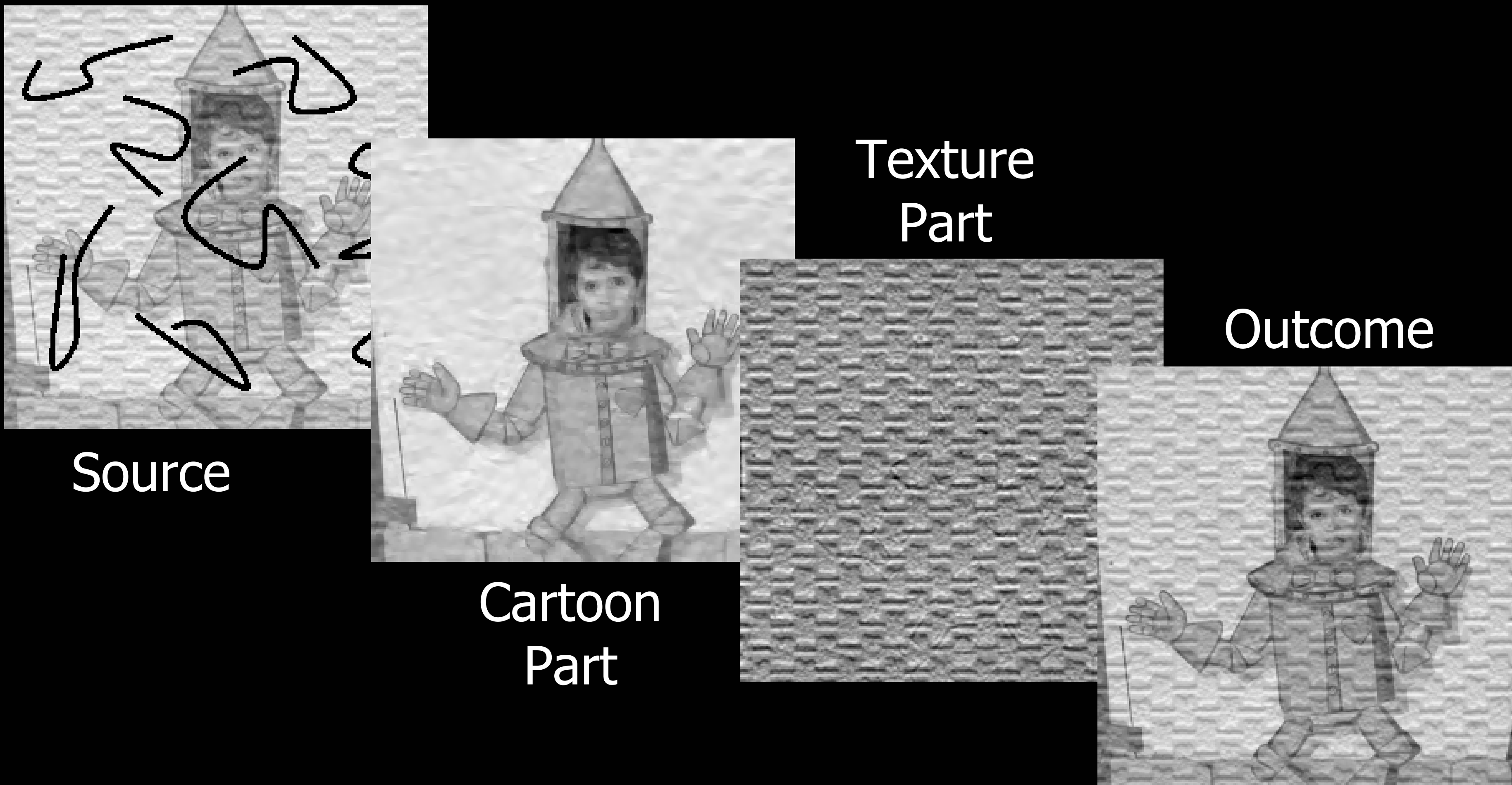
What if some values in s are unknown (with known locations!!!)?

$$\begin{bmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \end{bmatrix} = \underset{\underline{\alpha},\underline{\beta}}{\text{ArgMin}} \|\underline{\alpha}\|_1 + \|\underline{\beta}\|_1 + \lambda \|W(\underline{s} - \Phi_x \underline{\alpha} - \Phi_y \underline{\beta})\|_2^2$$

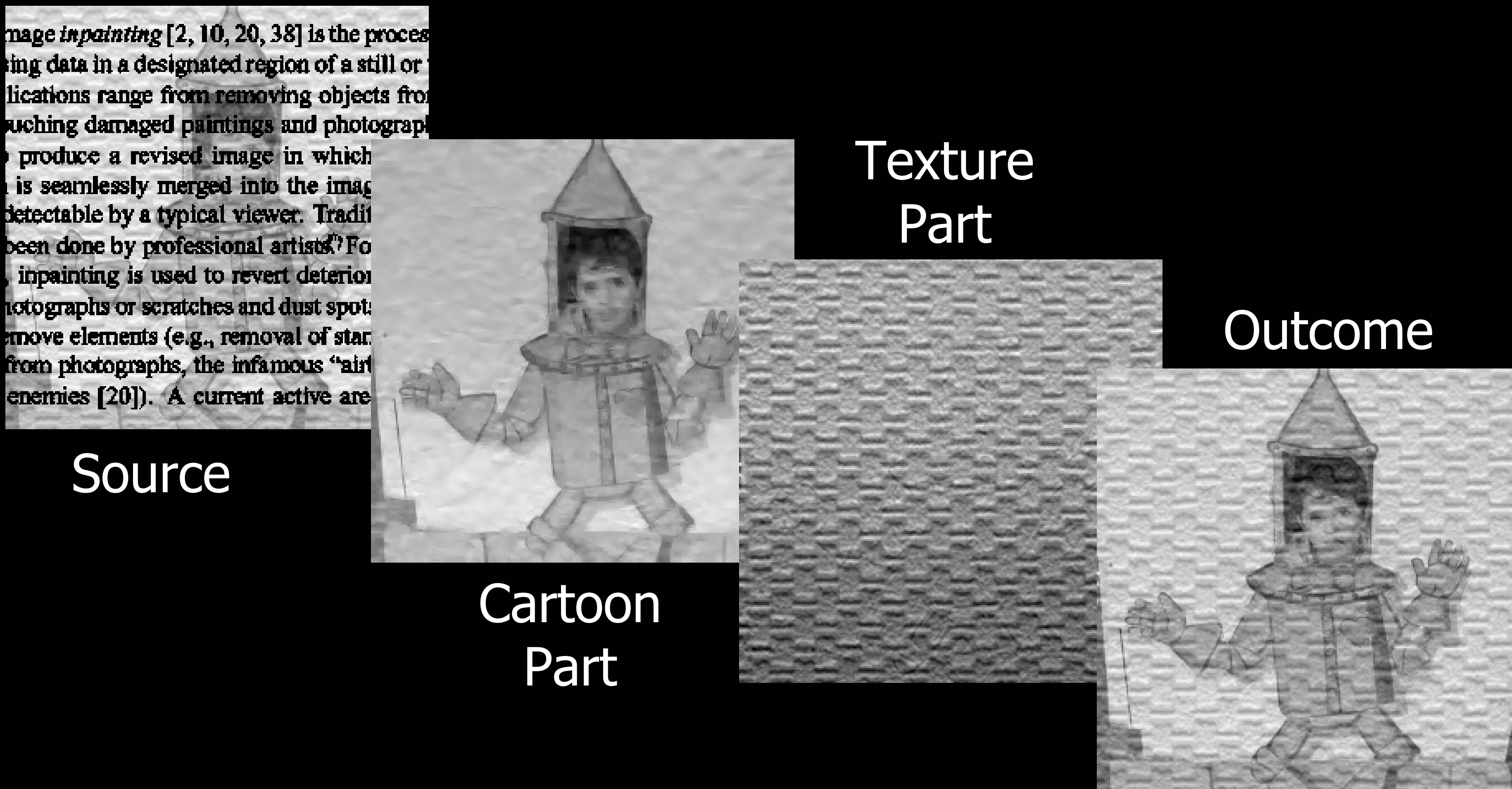
→ Noise removal
Inpainting
Decomposition

The image $\Phi_x \underline{\alpha} + \Phi_y \underline{\beta}$ will be the inpainted outcome.
Interesting comparison to Bertalmio et.al. ('02)

Results – Inpainting (1)



Results – Inpainting (2)



Summary

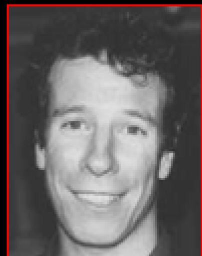
- Pursuit algorithms are successful as
  - Forward transform – we shed light on this behavior.
  - Regularization scheme in inverse problems – we have shown that the noiseless results extend nicely to treat this case as well.
- The dream: the over-completeness and sparsness ideas are highly effective, and should replace existing methods in signal representations and inverse-problems.
- We would like to contribute to this change by
  - Supplying clear(er) explanations about the BP/MP behavior,
  - Improve the involved numerical tools, and then
  - Deploy it to applications.

Future Work

- Many intriguing questions:
  - What dictionary to use? Relation to learning? SVM?
  - Improved bounds – average performance assessments?
  - Relaxed notion of sparsity? When zero is really zero?
  - How to speed-up BP solver (accurate/approximate)?
  - Applications – Coding? Restoration? …
- More information (including these slides) is found in **http://www.cs.technion.ac.il/~elad**

Some of the People Involved


Donoho, Stanford


Mallat, Paris


Coifman, Yale


Daubechies, Princetone


Temlyakov, USC


Gribonval, INRIA


Nielsen, Aalborg


Gilbert, Michigan


Tropp, Michigan


Strohmer, UC-Davis

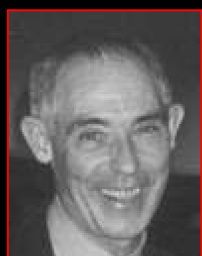

Candes, Caltech

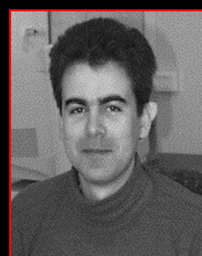

Romberg, CalTech


Tao, UCLA


Huo, GaTech


Rao, UCSD


Saunders, Stanford


Starck, Paris


Zibulevsky, Technion


Nemirovski, Technion


Feuer, Technion


Bruckstein, Technion