# Sparse Modeling

## in

## Image Processing and Deep Learning

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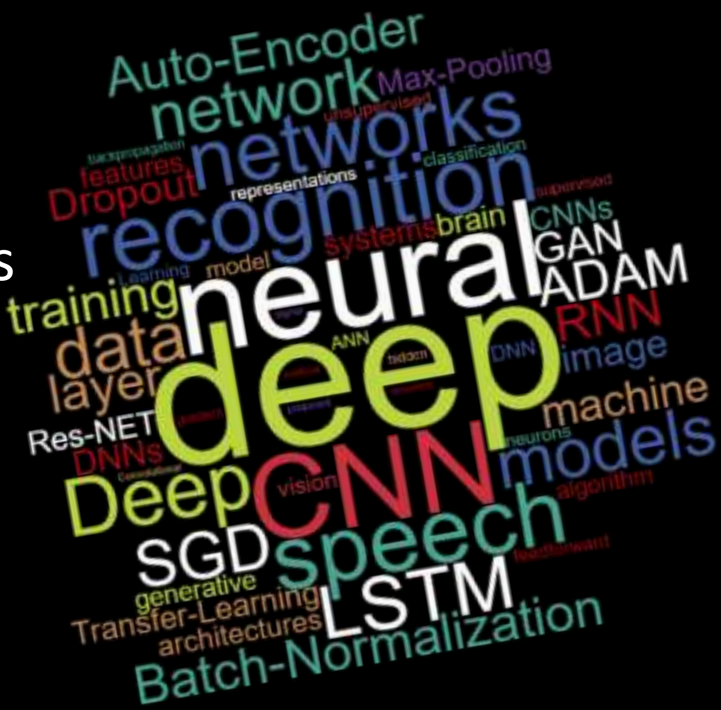
# This Lecture is About …

A Proposed Theory for Deep-Learning (DL)

Explanation:

- DL has been extremely successful in solving a variety of learning problems
- DL is an empirical field, with numerous tricks and know-how, but almost no theoretical foundations
- A theory for DL has become the holy-grail of current research in Machine-Learning and related fields

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# Who Needs Theory ?

**We All Do !!**

… because … A theory

- … could bring the next rounds of ideas to this field, breaking existing barriers and opening new opportunities
- … could map clearly the limitations of existing DL solutions, and point to key features that control their performance
- … could remove the feeling with many of us that DL is a "dark magic", turning it into a solid scientific discipline

Ali Rahimi: NIPS 2017 Test-of-Time Award

"Machine learning has become alchemy"

Yan LeCun

Understanding is a good thing … but another goal is inventing methods. In the history of science and technology, engineering preceded theoretical understanding:

- Lens & telescope → Optics
- Steam engine → Thermodynamics
- Airplane → Aerodynamics
- Radio & Comm. → Info. Theory
- Computer → Computer Science

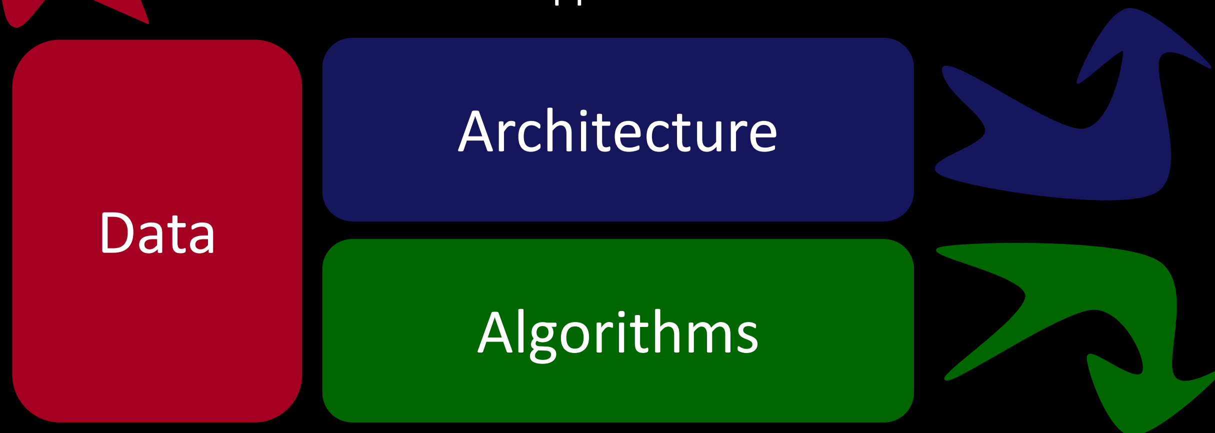
# A Theory for DL ?

Stephane Mallat (ENS) & Joan Bruna (NYU): Proposed the scattering transform (wavelet-based) and emphasized the treatment of invariances in the input data

Richard Baraniuk & Ankit Patel (RICE): Offered a generative probabilistic model for the data, showing that classic architectures relate to it

Raja Giryes (TAU): Studied the architecture of DNN in the context of their ability to give distance-preserving embedding of signals

Gitta Kutyniok (TU) & Helmut Bolcskei (ETH): Studied the ability of DNN architectures to approximate families of functions




Rene Vidal (JHU): Explained the ability to optimize the typical non-convex objective and yet get to a global minima

Naftali Tishby (HUJI): Introduced the Information Bottleneck (IB) concept and demonstrated its relevance to deep learning

Stefano Soatto's team (UCLA): Analyzed the Stochastic Gradient Descent (SGD) algorithm, connecting it to the IB objective

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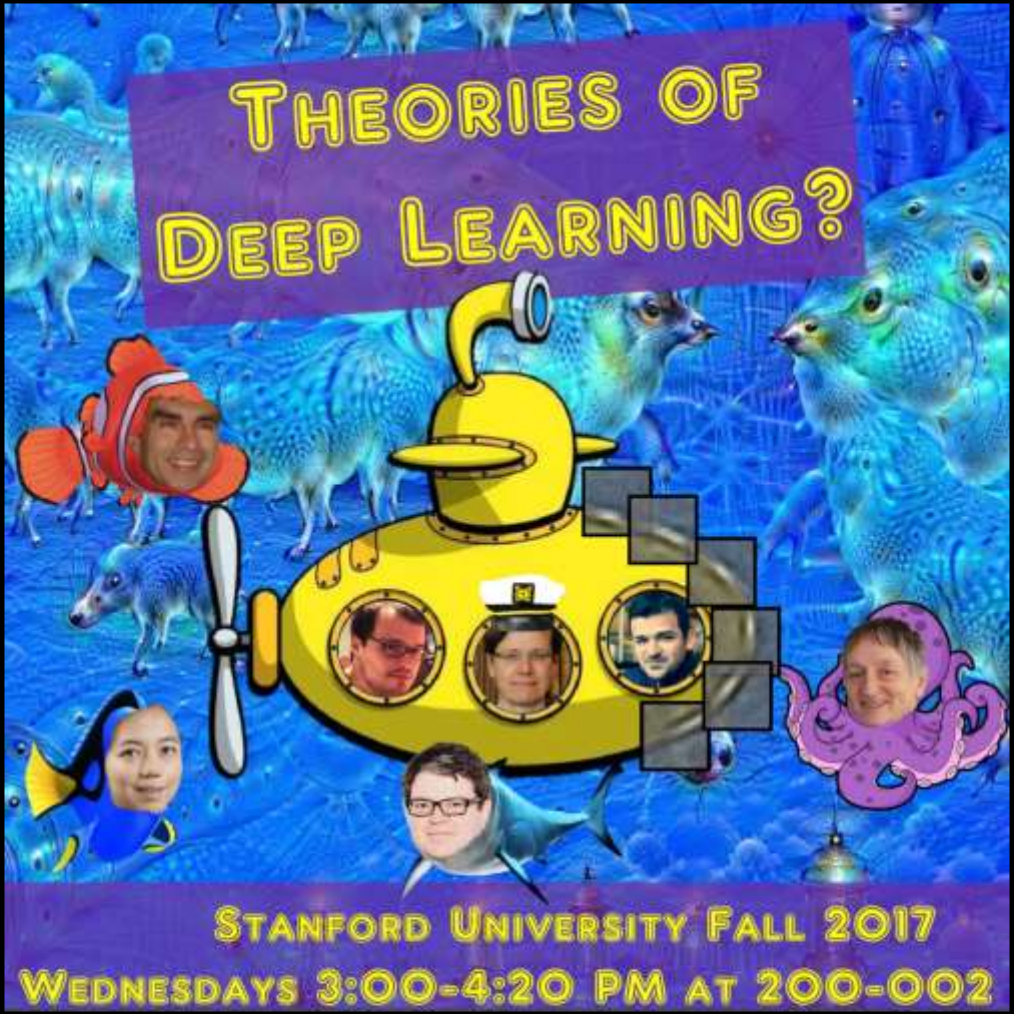
# So, is there a Theory for DL ?




The answer is tricky:

There are already various such attempts, and some of them are truly impressive

… but …

none of them is complete

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# Interesting Observations

- Theory origins: Signal Proc., Control Theory, Info. Theory, Harmonic Analysis, Sparse Represen., Quantum Physics, PDE, Machine learning …

Ron Kimmel: "*DL is a dark monster covered with mirrors. Everyone sees his reflection in it …*"

David Donoho: "*… these mirrors are taken from Cinderella's story, telling each that he is the most beautiful*"

√ Data

√ Architecture

Algorithms

- Today's talk is on our proposed theory:

Yaniv Romano

Vardan Papyan

Jeremias Sulam

… and yes, our theory is the best

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# This Lecture: More Specifically

**Sparseland**
Sparse Representation Theory

**CSC**
Convolutional Sparse Coding

**ML-CSC**
Multi-Layered Convolutional Sparse Coding

Sparsity-Inspired Models → Deep-Learning

Another underlying idea that accompanies us

Generative modeling of data sources enables

- A systematic algorithm development, &
- A theoretical analysis of their performance

**Disclaimer:** Being a lecture on the theory of DL, this lecture is … theoretical … and mathematically oriented

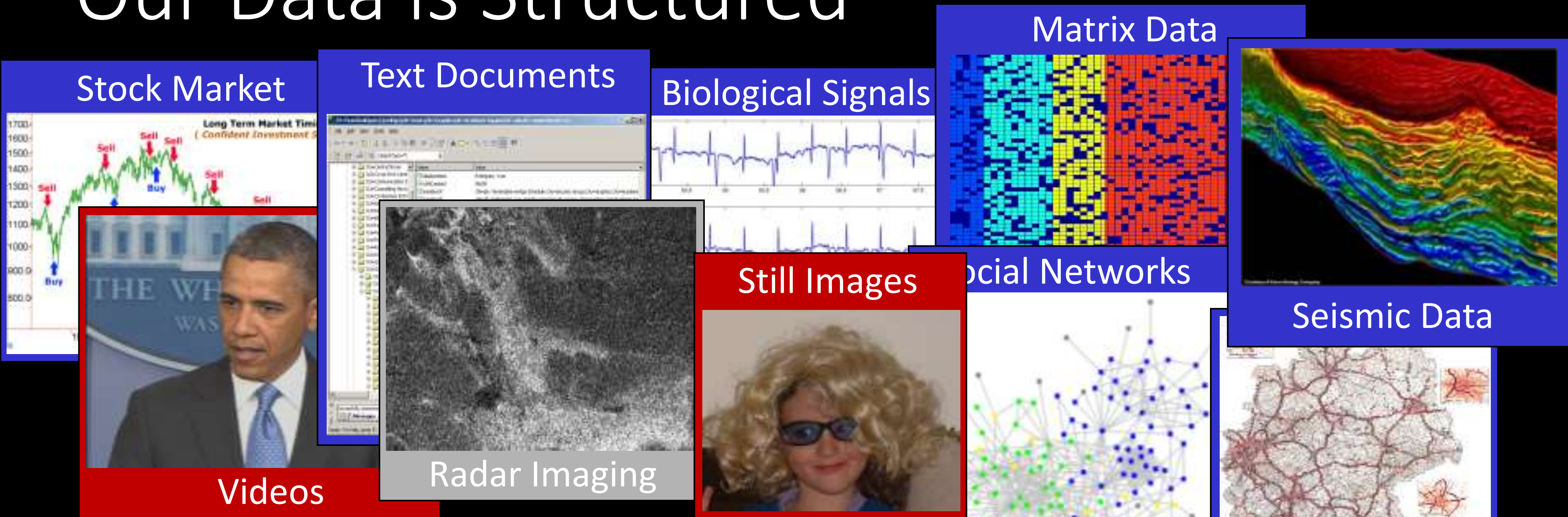
Our eventual goal in today's talk is to present the …

# Multi-Layered Convolutional Sparse Modeling

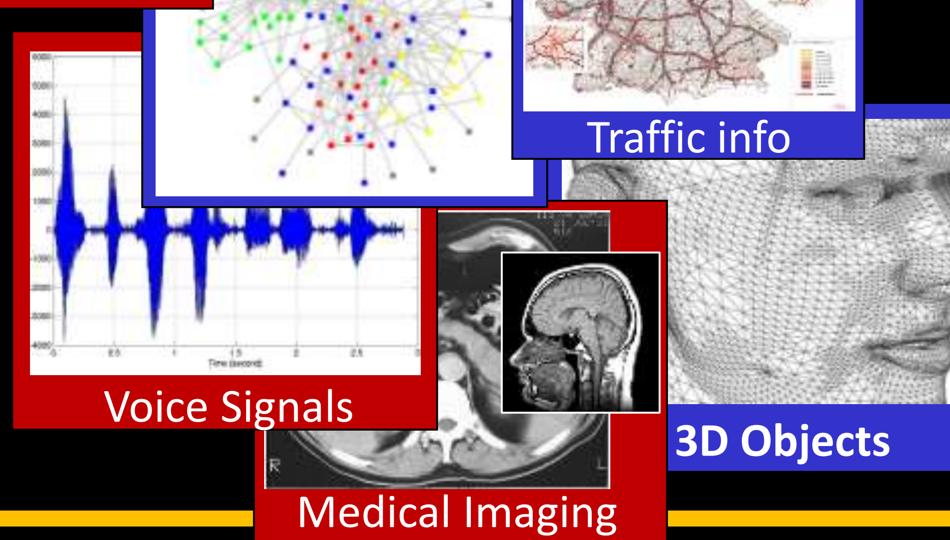
So, lets use this as our running title,
parse it into words,
and explain each of them

# Multi-Layered Convolutional Sparse Modeling

# Our Data is Structured




- We are surrounded by various diverse sources of massive information
- Each of these sources have an internal structure, which can be exploited
- This structure, when identified, is the engine behind the ability to process data
- How to identify structure?



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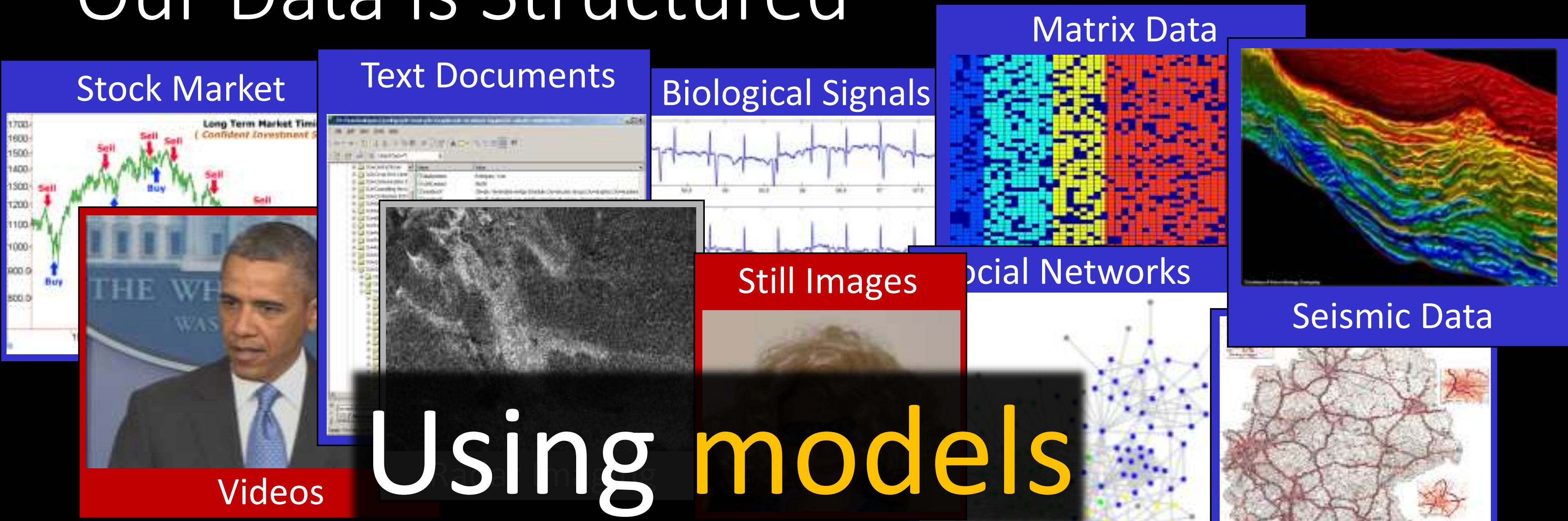


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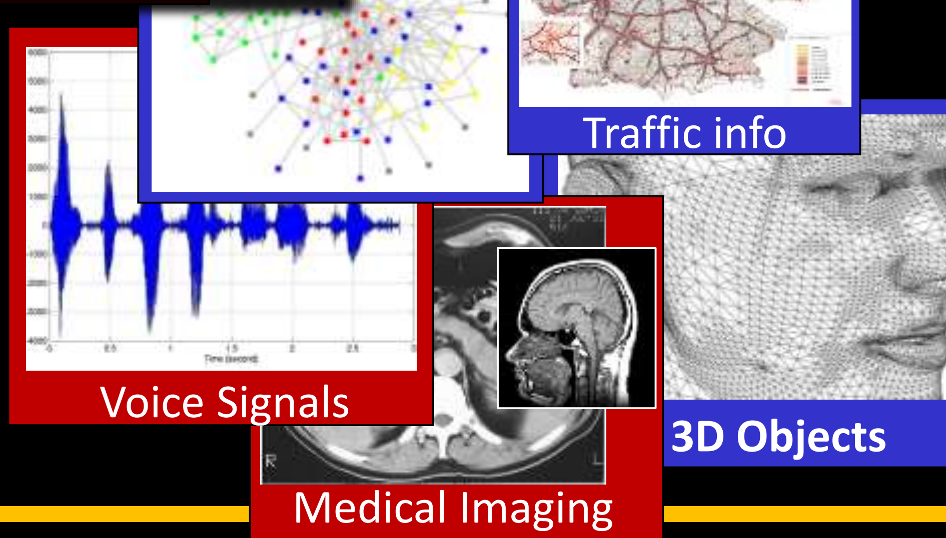
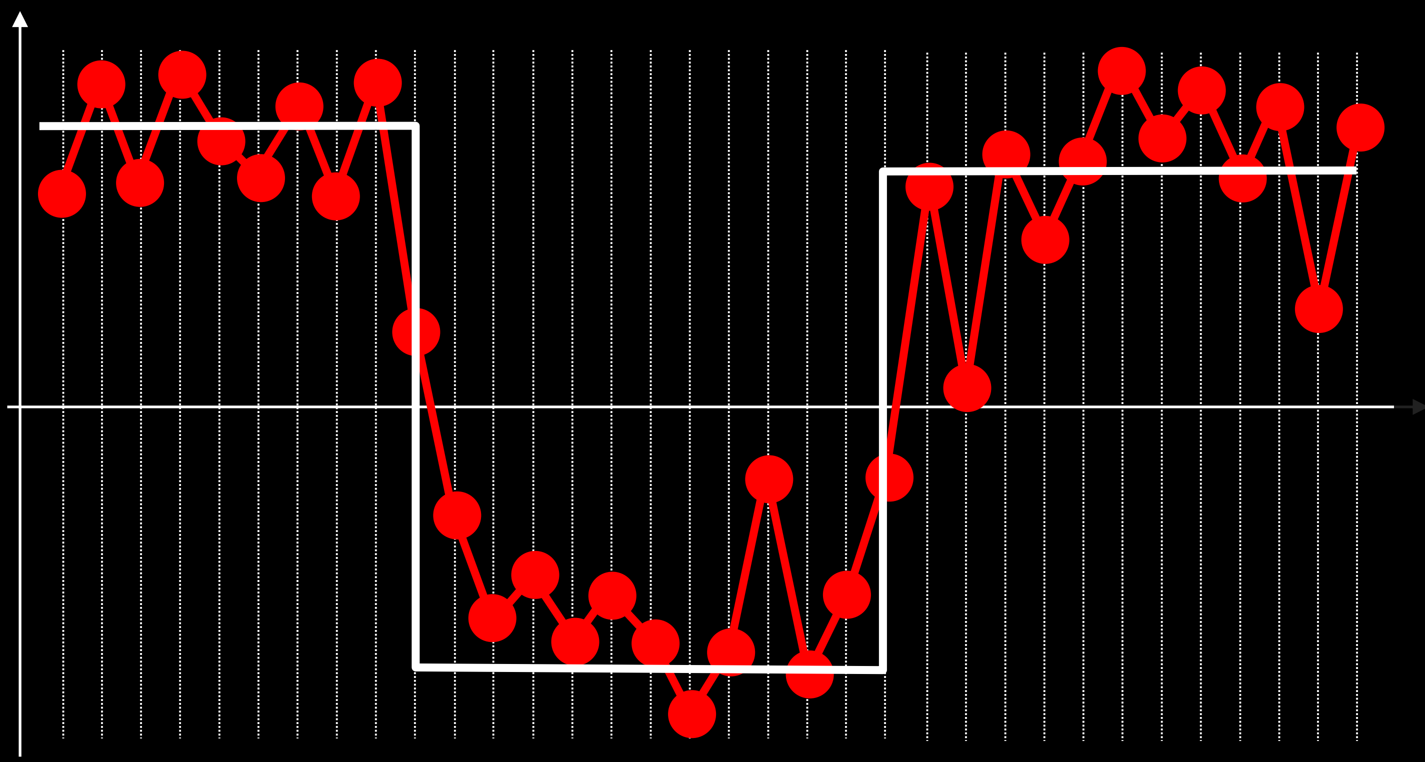


# Model?



**Fact 1**:
This signal contains AWGN $\mathbb{N}(0,1)$

**Fact 2**:
The clean signal is believed to be PWC

Effective removal of noise (and many other tasks) relies on an proper modeling of the signal

# Models

- A model: a mathematical description of the underlying signal of interest, describing our beliefs regarding its structure
- The following is a partial list of commonly used models for images
- Good models should be simple while matching the signals

Simplicity ⟷ Reliability

- Models are almost always imperfect

Principal-Component-Analysis
Gaussian-Mixture
Markov Random Field
Laplacian Smoothness
DCT concentration
Wavelet Sparsity
Piece-Wise-Smoothness
C2-smoothness
Besov-Spaces
Total-Variation
Beltrami-Flow

# What this Talk is all About?

## Data Models and Their Use

- Almost any task in data processing requires a model – true for denoising, deblurring, super-resolution, inpainting, compression, anomaly-detection, sampling, recognition, separation, and more
- Sparse and Redundant Representations offer a new and highly effective model – we call it

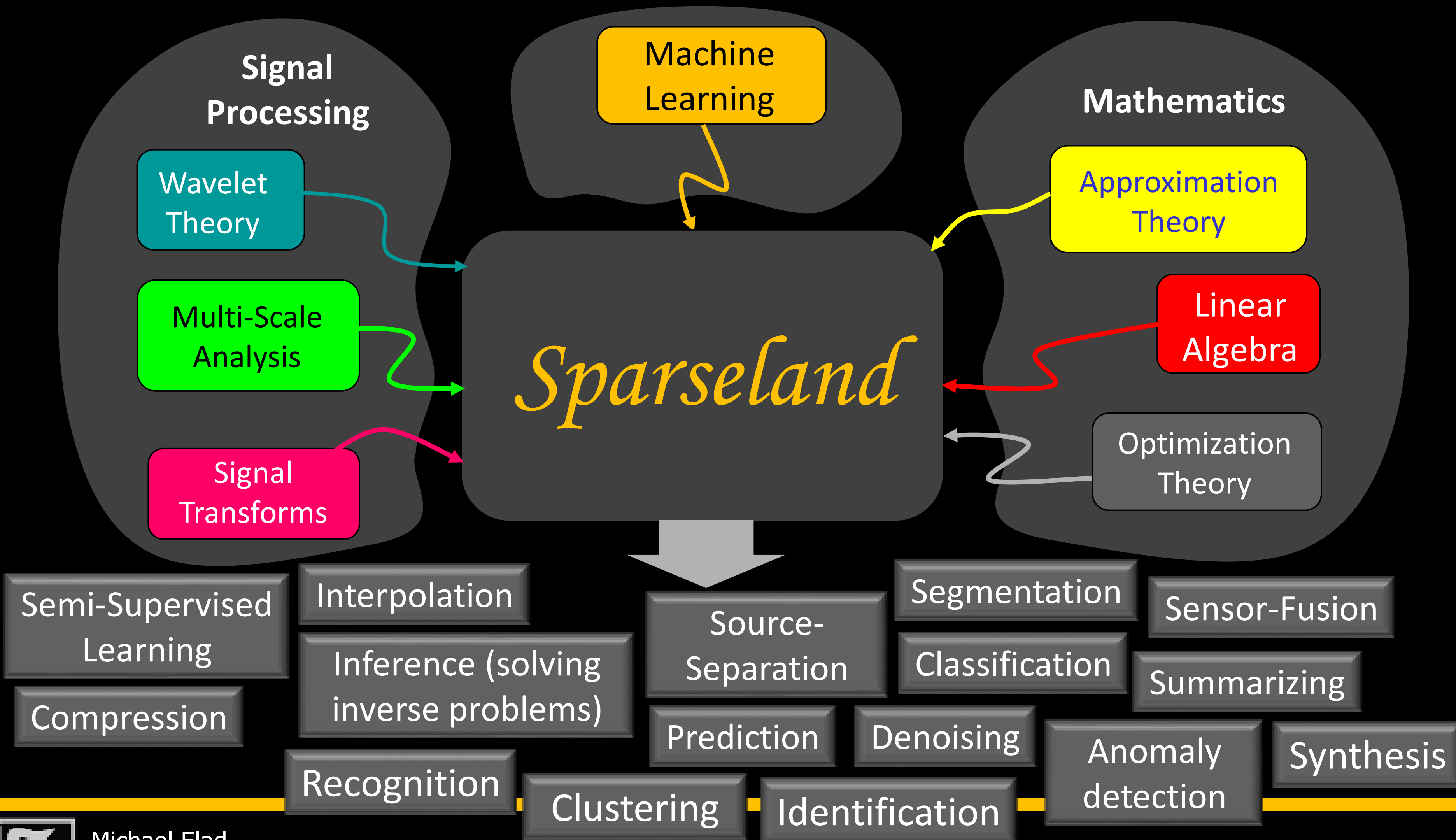
  *Sparseland*

- We shall describe this and descendant versions of it that lead all the way to … deep-learning

# Multi-Layered Convolutional

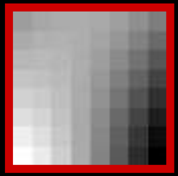
# Sparse Modeling

# A New Emerging Model

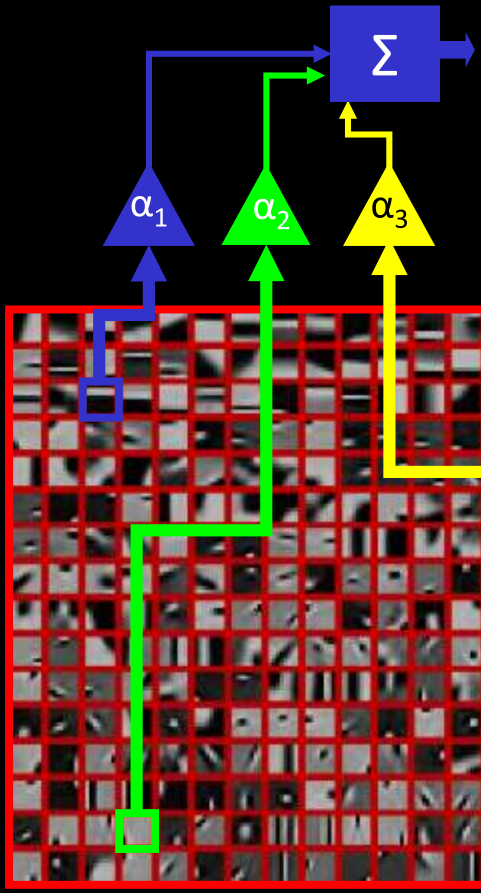


# The *Sparseland* Model

- Task: model image patches of size 8×8 pixels



- We assume that a **dictionary** of such image patches is given, containing 256 **atom** images




- The *Sparseland* model assumption: every image patch can be described as a linear combination of **few** atoms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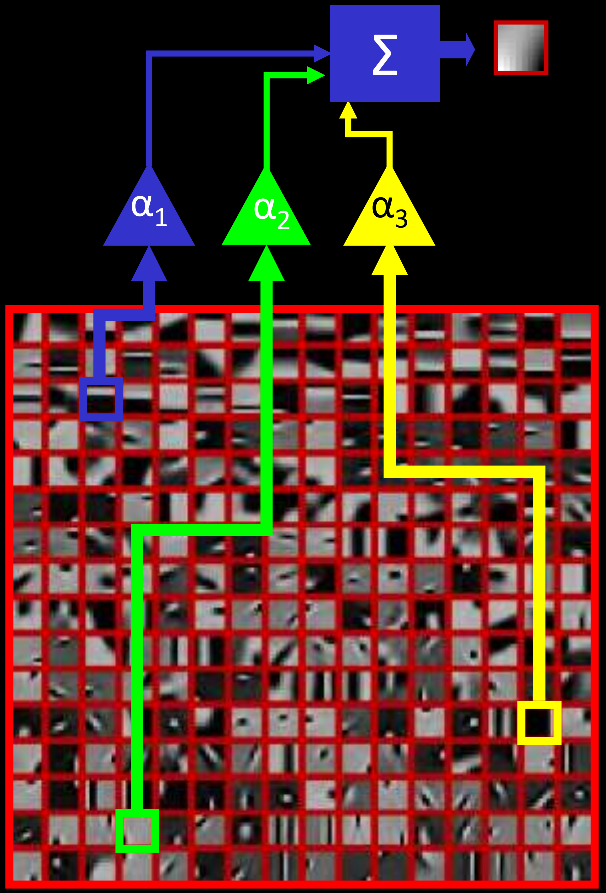
# The *Sparseland* Model

Properties of this model:

**Sparsity** and **Redundancy**

- We start with a 8-by-8 pixels patch and represent it using 256 numbers
  – This is a redundant representation
- However, out of those 256 elements in the representation, only 3 are non-zeros
  – This is a sparse representation
- Bottom line in this case: 64 numbers representing the patch are replaced by 6 (3 for the indices of the non-zeros, and 3 for their entries)





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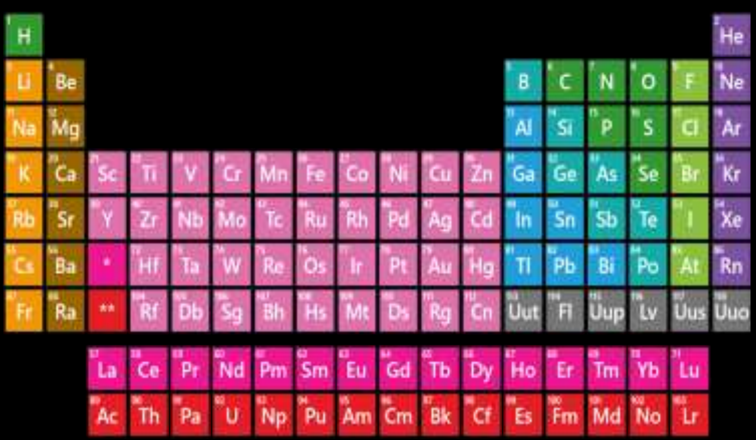
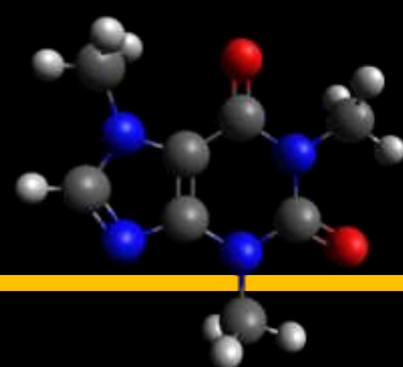
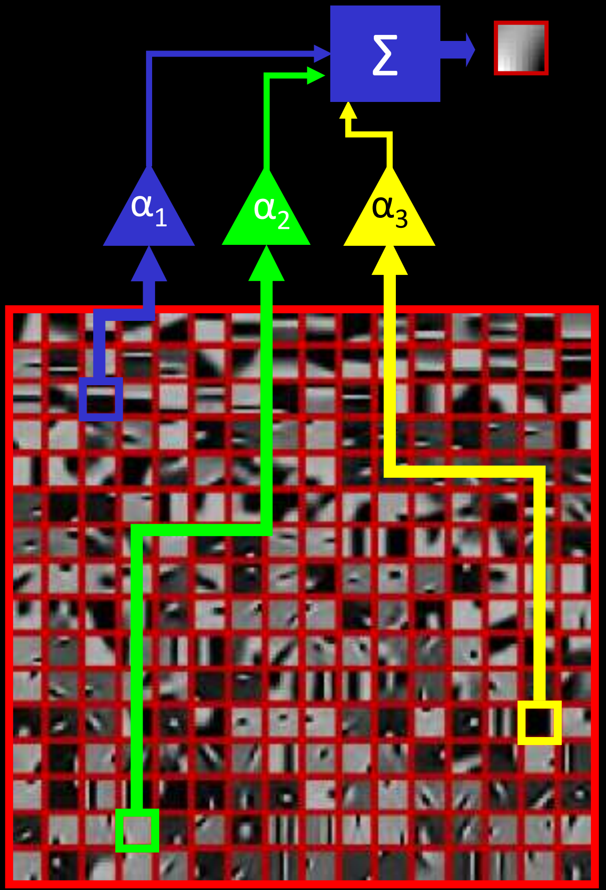
# Chemistry of Data

We could refer to the *Sparseland* model as the chemistry of information:

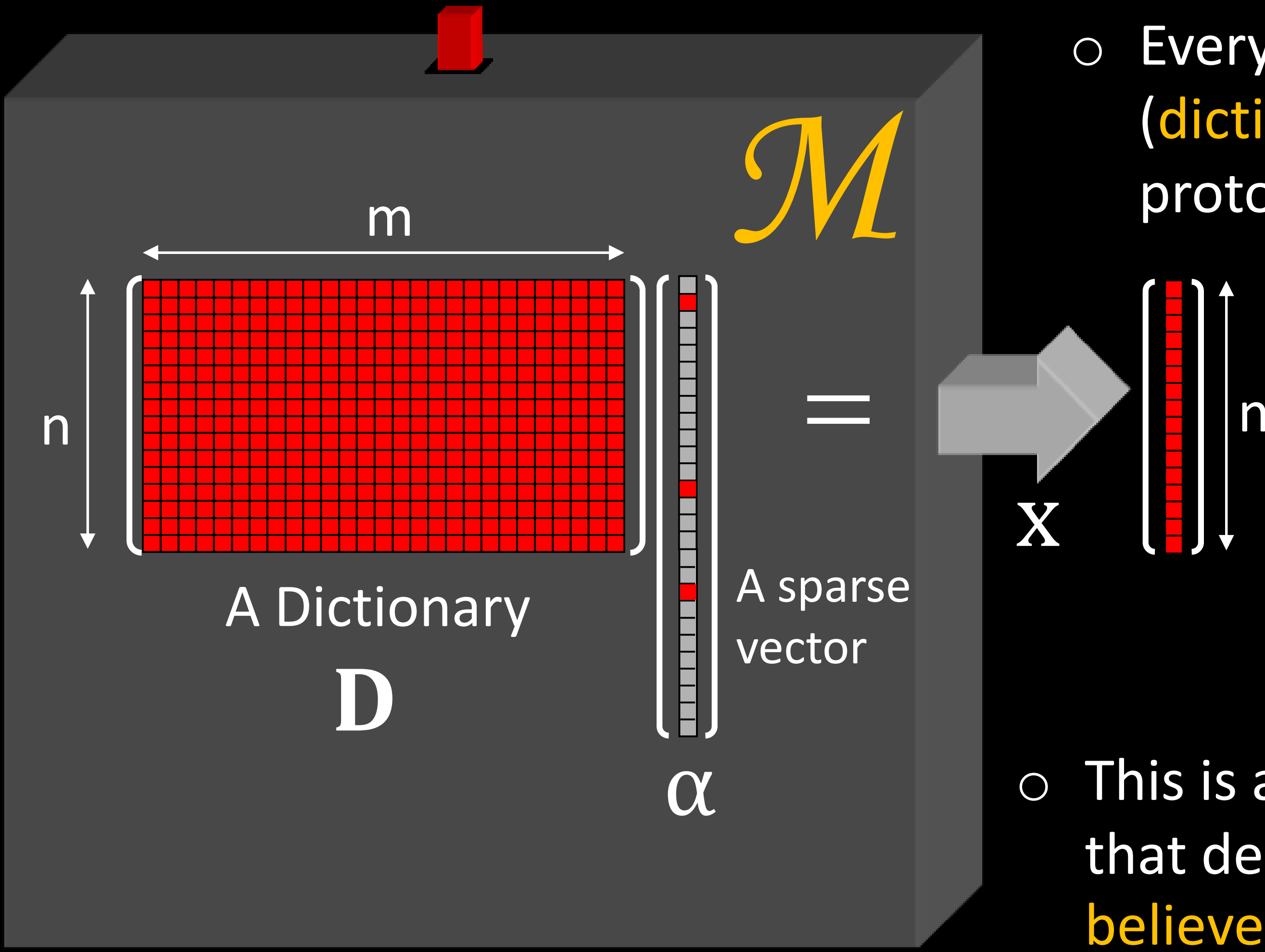
- Our dictionary stands for the Periodic Table containing all the elements
- Our model follows a similar rationale: Every molecule is built of **few** elements





# *Sparseland* : A Formal Description



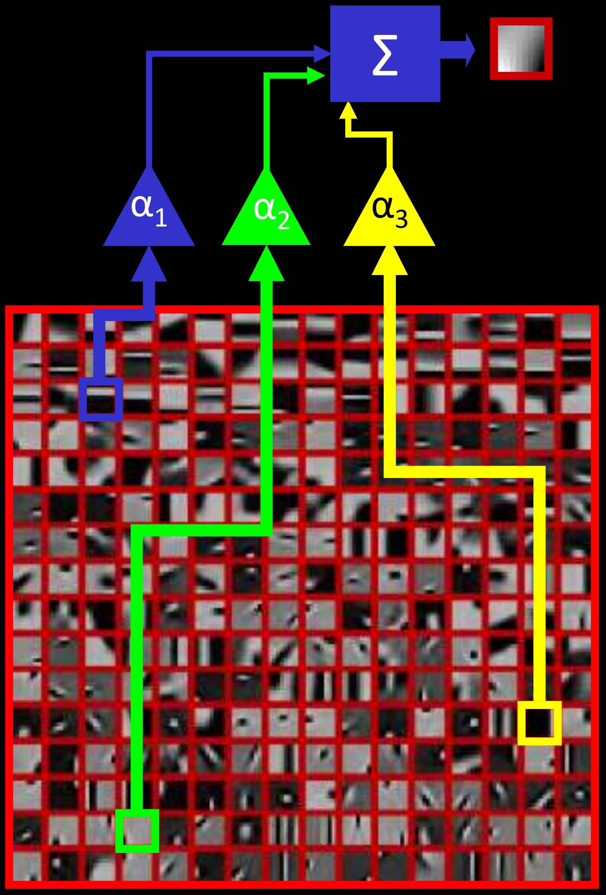

- Every column in **D** (dictionary) is a prototype signal (atom)
  - The vector $\underline{\alpha}$ is generated with few non-zeros at arbitrary locations and values
- This is a generative model that describes how (we believe) signals are created

# Difficulties with *Sparseland*

- Problem 1: Given a signal, how can we find its atom decomposition?
- A simple example:
  - There are 2000 atoms in the dictionary
  - The signal is known to be built of 15 atoms

$$\Rightarrow \binom{2000}{15} \approx 2.4e+37 \quad \text{possibilities}$$

  - If each of these takes 1nano-sec to test, will take ~7.5e20 years to finish !!!!!!
- So, are we stuck?





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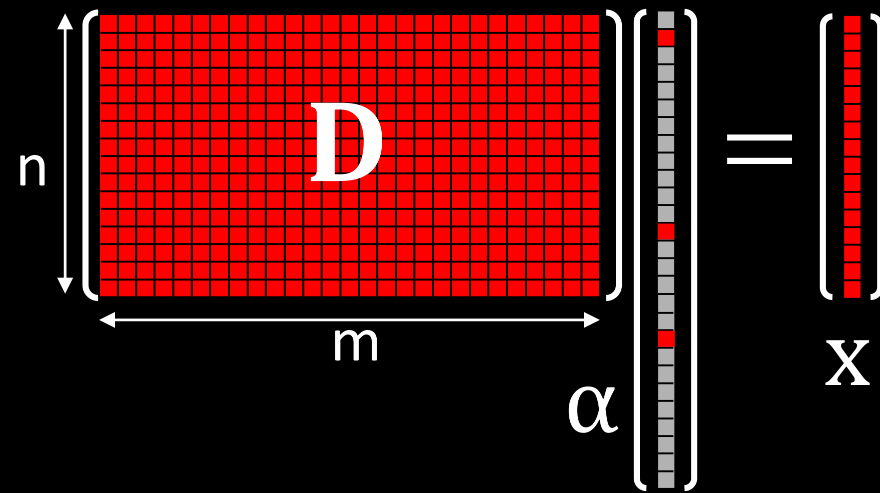
# Atom Decomposition Made Formal

$$\min_{\alpha} \|\alpha\|_0 \;\; \text{s.t.} \;\; \mathbf{x} = \mathbf{D}\alpha$$

$$\min_{\alpha} \|\alpha\|_0 \;\; \text{s.t.} \;\; \|\mathbf{D}\alpha - \mathbf{y}\|_2 \leq \varepsilon$$




Approximation Algorithms

Relaxation methods

Basis-Pursuit

Greedy methods

Thresholding/OMP

- $L_0$ – counting number of non-zeros in the vector
- This is a projection onto the *Sparseland* model
- These problems are known to be NP-Hard problem

# Pursuit Algorithms

$$\min_\alpha \|\alpha\|_0 \;\; \text{s.t.} \;\; \|\mathbf{D}\alpha - \mathrm{y}\|_2 \le \varepsilon$$

Approximation Algorithms

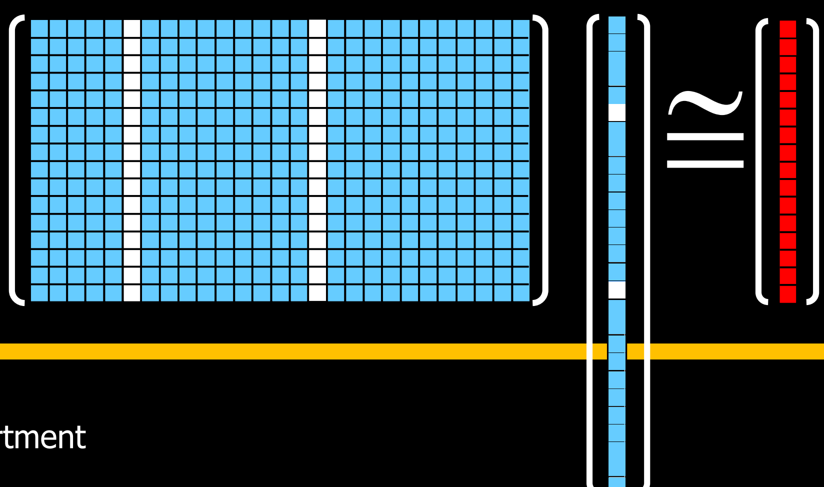
## Basis Pursuit

Change the $L_0$ into $L_1$ and then the problem becomes convex and manageable

$$\min_\alpha \|\alpha\|_1 \;\; \text{s.t.} \;\; \|\mathbf{D}\alpha - \mathrm{y}\|_2 \le \varepsilon$$

## Matching Pursuit

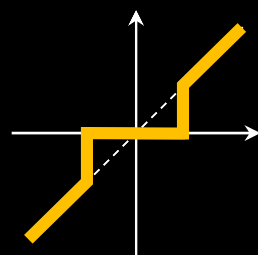
Find the support greedily, one element at a time



## Thresholding

Multiply $\mathrm{y}$ by $\mathbf{D}^{\mathbf{T}}$ and apply shrinkage:

$$\hat{\alpha} = \mathcal{P}_\beta\{\mathbf{D}^{\mathbf{T}}\mathrm{y}\}$$



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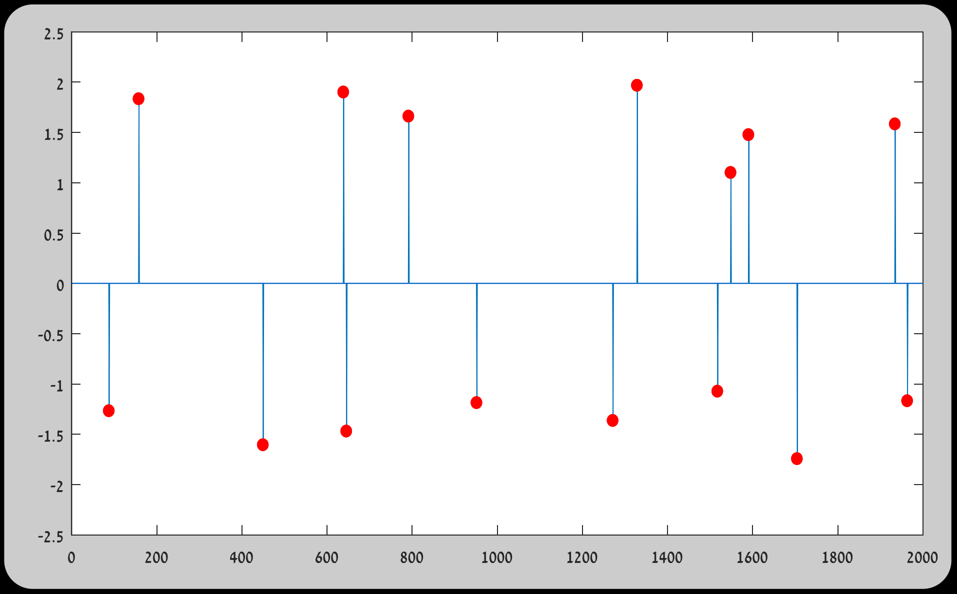
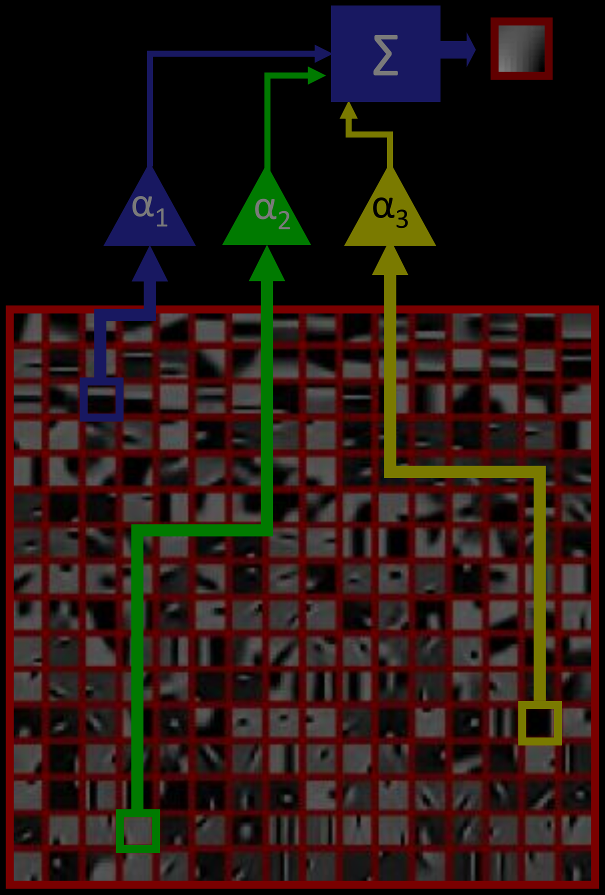
# Difficulties with *Sparseland*

- There are various pursuit algorithms
- Here is an example using the Basis Pursuit ($L_1$):




- Surprising fact: Many of these algorithms are often accompanied by theoretical guarantees for their success, if the unknown is sparse enough

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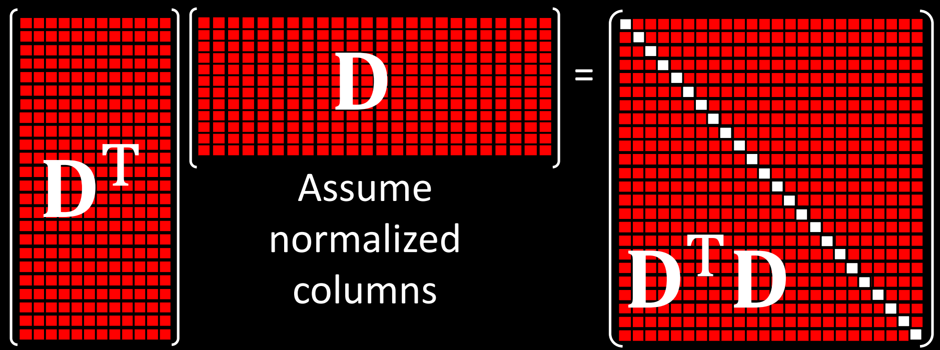
# The Mutual Coherence

- Compute




- The Mutual Coherence $\mu(\mathbf{D})$ is the largest off-diagonal entry in absolute value
- We will pose all the theoretical results in this talk using this property, due to its simplicity
- You may have heard of other ways to characterize the dictionary (Restricted Isometry Property - RIP, Exact Recovery Condition - ERC, Babel function, Spark, …)

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# Basis-Pursuit Success

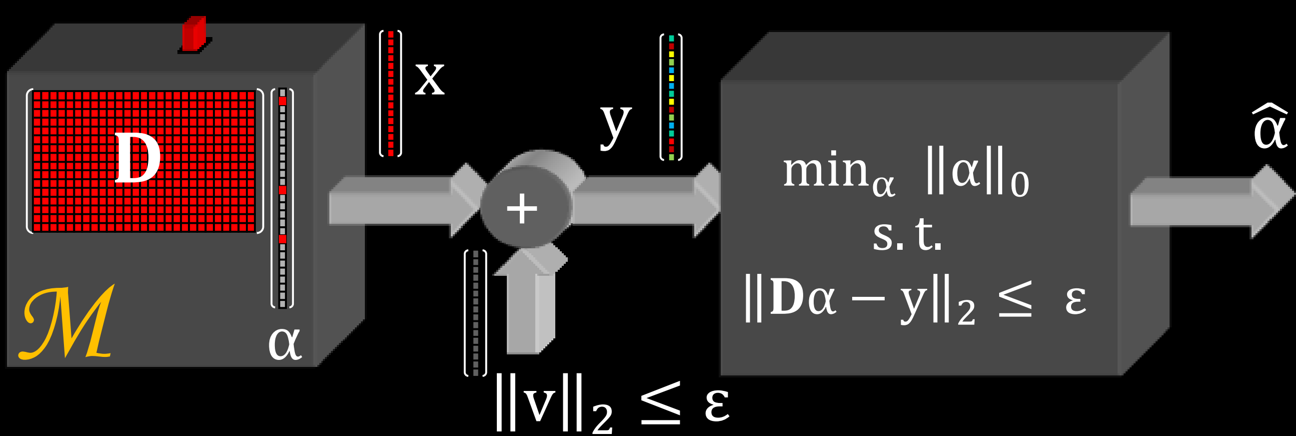
**Theorem: Given** a noisy signal $y = \mathbf{D}\alpha + v$ where $\|v\|_2 \le \varepsilon$ and $\alpha$ is sufficiently sparse,

$$\|\alpha\|_0 < \frac{1}{4}\left(1 + \frac{1}{\mu}\right)$$

**then Basis-Pursuit**: $\min_\alpha \ \|\alpha\|_1 \ \text{ s.t. } \ \|\mathbf{D}\alpha - y\|_2 \le \varepsilon$

**leads to a stable result**: $\|\hat{\alpha} - \alpha\|_2^2 \le \dfrac{4\varepsilon^2}{1-\mu(4\|\alpha\|_0-1)}$

Donoho, Elad & Temlyakov ('06)

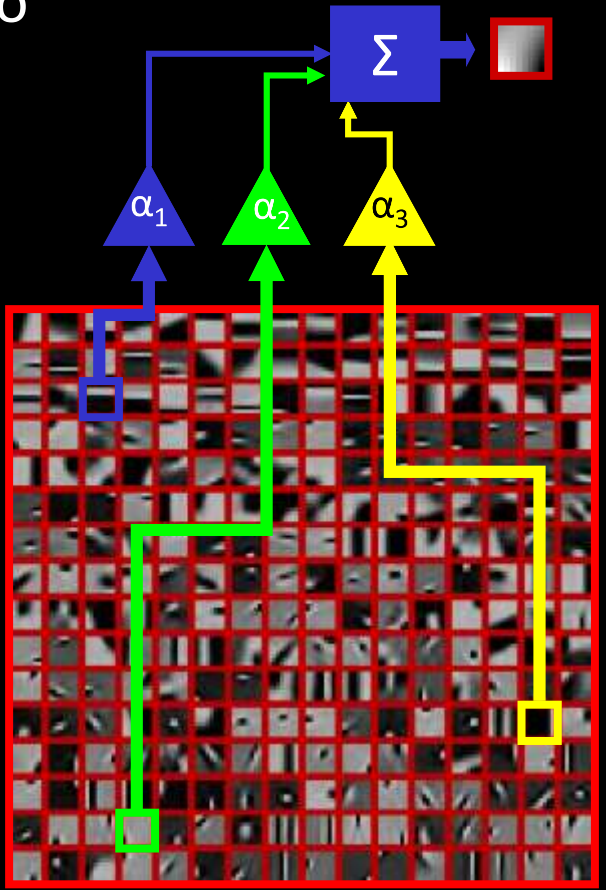



Comments:

- If $\varepsilon$=0 → $\hat{\alpha} = \alpha$
- This is a worst-case analysis – better bounds exist
- Similar theorems exist for many other pursuit algorithms

# Difficulties with *Sparseland*

- Problem 2: Given a family of signals, how do we find the dictionary to represent it well?
- Solution: Learn! Gather a large set of signals (many thousands), and find the dictionary that sparsifies them
- Such algorithms were developed in the past 10 years (e.g., K-SVD), and their performance is surprisingly good
- We will not discuss this matter further in this talk due to lack of time

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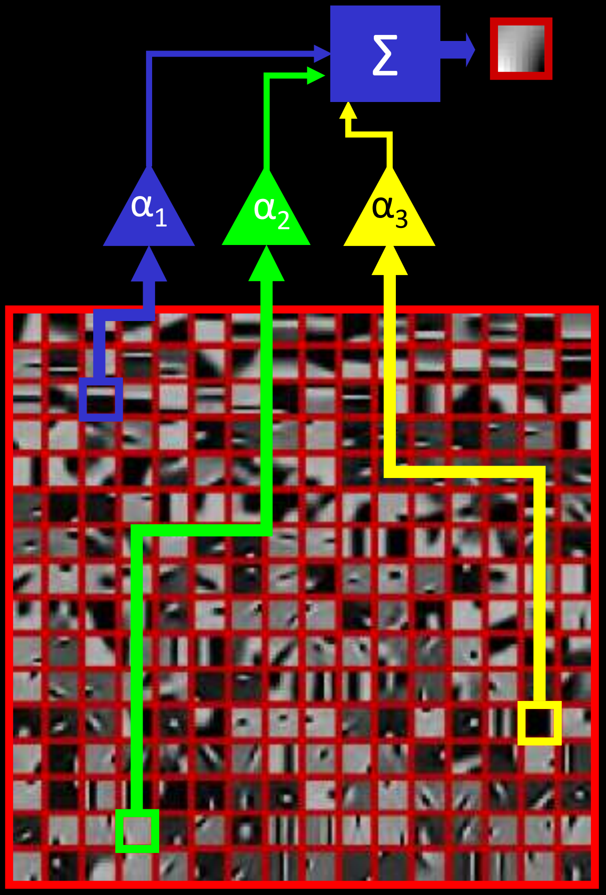
# Difficulties with *Sparseland*

- Problem 3: Why is this model suitable to describe various sources? e.g., Is it good for images? Audio? Stocks? …

- General answer: Yes, this model is extremely effective in representing various sources
  - Theoretical answer: Clear connection to other models
  - Empirical answer: In a large variety of signal and image processing (and later machine learning), this model has been shown to lead to state-of-the-art results





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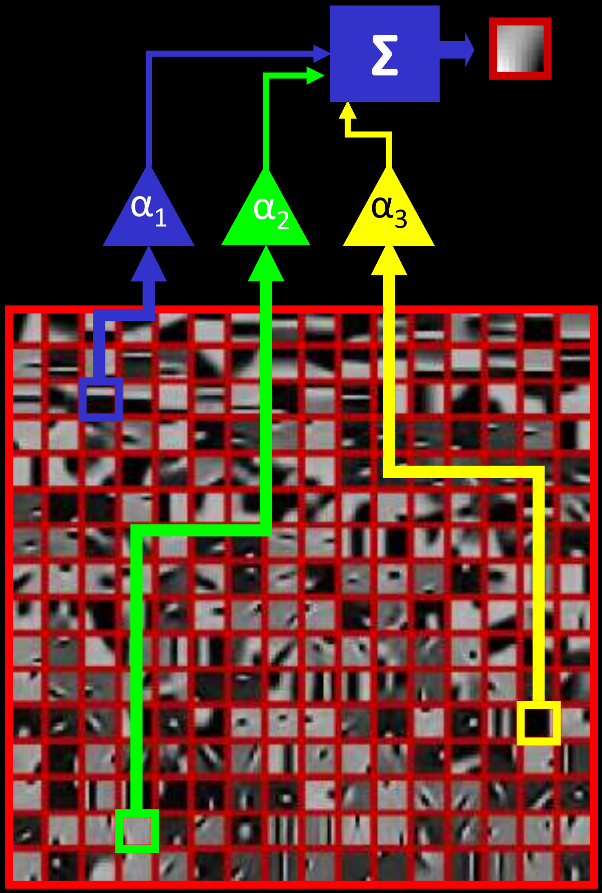
# Difficulties with *Sparseland* ?

- Problem 1: Given an image patch, how can we find its atom decomposition ?
- Problem 2: Given a family of signals, how do we find the dictionary to represent it well?
- Problem 3: Is this model flexible enough to describe various sources? E.g., Is it good for images? audio? …



# Difficulties with *Sparseland* ?

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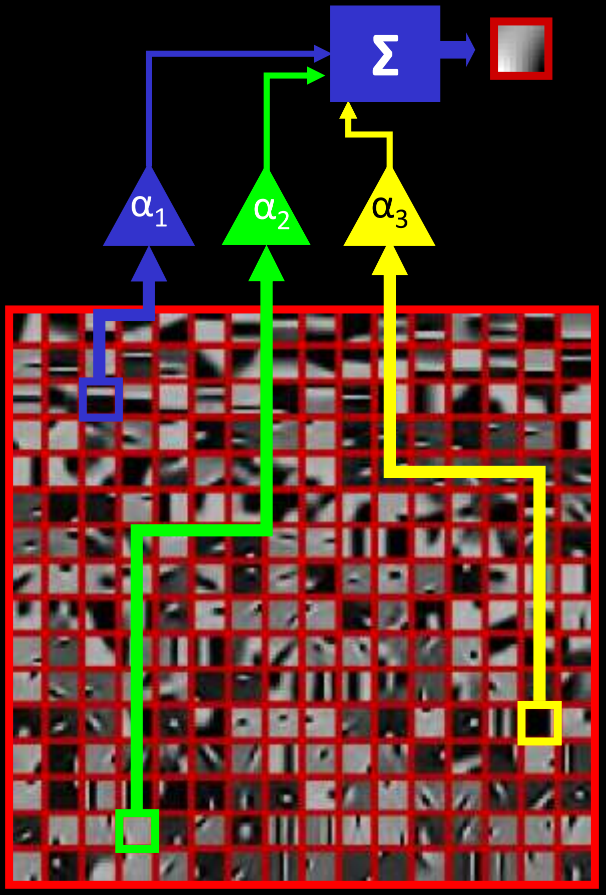
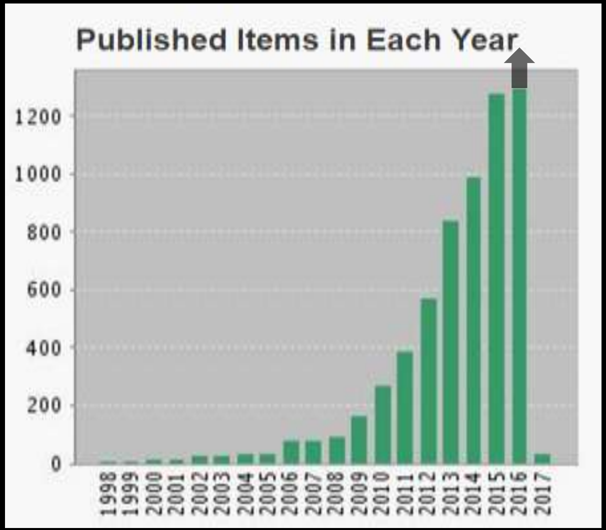
**ALL ANSWERED POSITIVELY AND CONSTRUCTIVELY**



# This Field has been rapidly GROWING …

- *Sparseland* has a great success in signal & image processing and machine learning tasks
- In the past 8-9 years, many books were published on this and closely related fields





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Rama Chellappa
Sparse Representations and Compressive Sensing for Imaging and Vision
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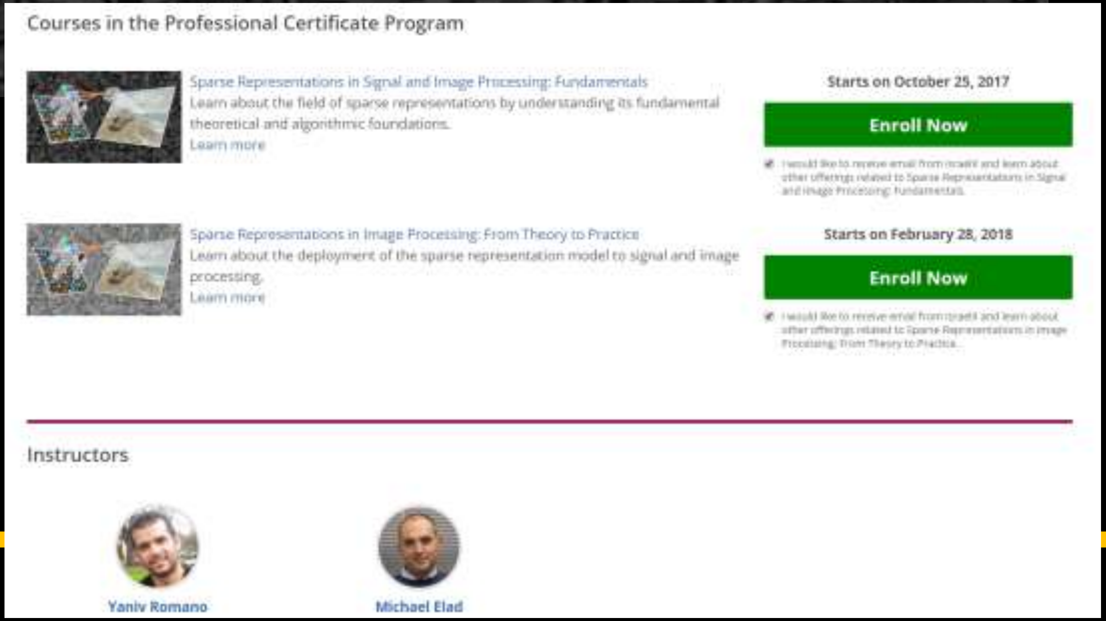
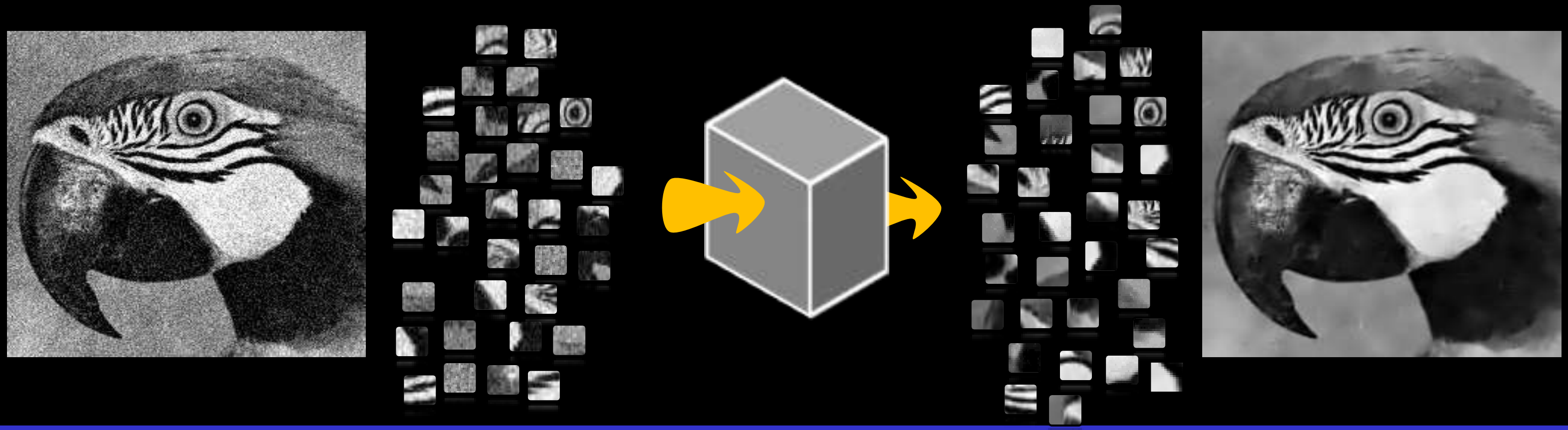


# *Sparseland* for Image Processing

- When handling images, *Sparseland* is typically deployed on small overlapping patches due to the desire to train the model to fit the data better



- The model assumption is: each patch in the image is believed to have a sparse representation w.r.t. a common local dictionary
- What is the corresponding global model? This brings us to … the Convolutional Sparse Coding (CSC)

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# Multi-Layered Convolutional Sparse Modeling

Joint work with

Yaniv Romano

Vardan Papyan

Jeremias Sulam

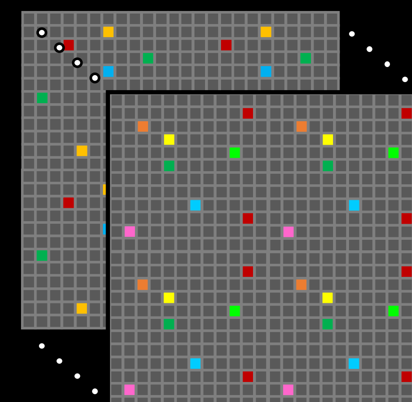
# Convolutional Sparse Coding (CSC)

$m$ filters convolved with their sparse representations

i-th feature-map: An image of the same size as $\mathbf{X}$ holding the sparse representation related to the i-filter

An image with $N$ pixels

$$[\mathbf{X}] = \sum_{i=1}^{m} \mathrm{d}_i * [\Gamma_i]$$



The i-th filter of small size $n$

This model emerged in 2005-2010, developed and advocated by Yan LeCun and others. It serves as the foundation of Convolutional Neural Networks

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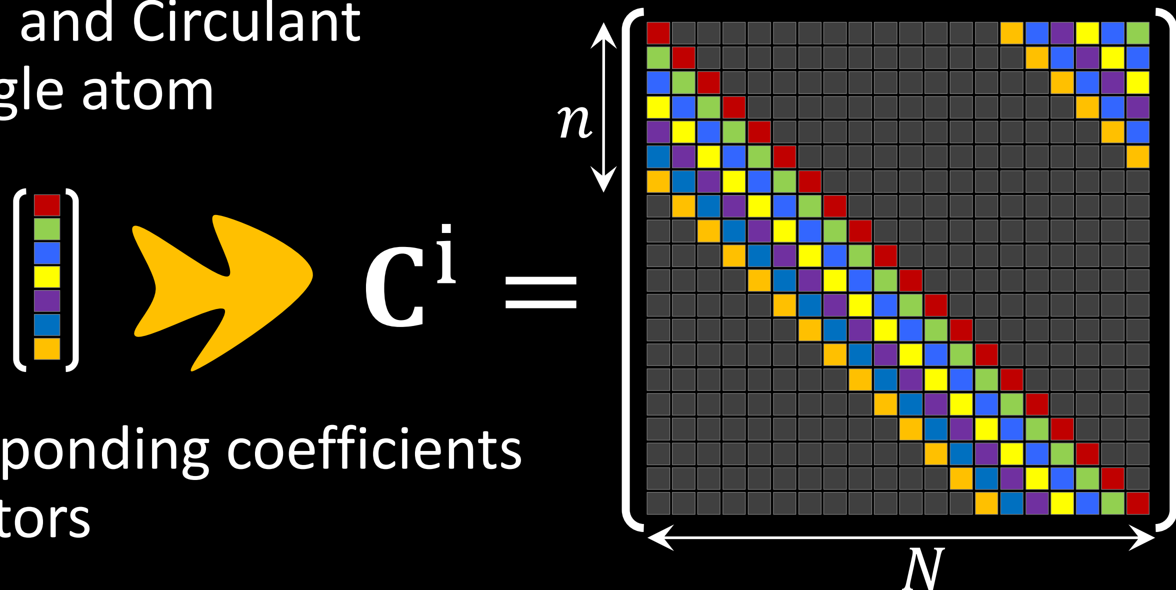
# CSC in Matrix Form

- Here is an alternative global sparsity-based model formulation

$$\mathbf{X} = \sum_{\mathrm{i}=1}^{m} \mathbf{C}^{\mathrm{i}} \mathbf{\Gamma}^{\mathrm{i}} = [\mathbf{C}^1 \cdots \mathbf{C}^m] \begin{bmatrix} \mathbf{\Gamma}^1 \\ \vdots \\ \mathbf{\Gamma}^m \end{bmatrix} = \mathbf{D}\mathbf{\Gamma}$$

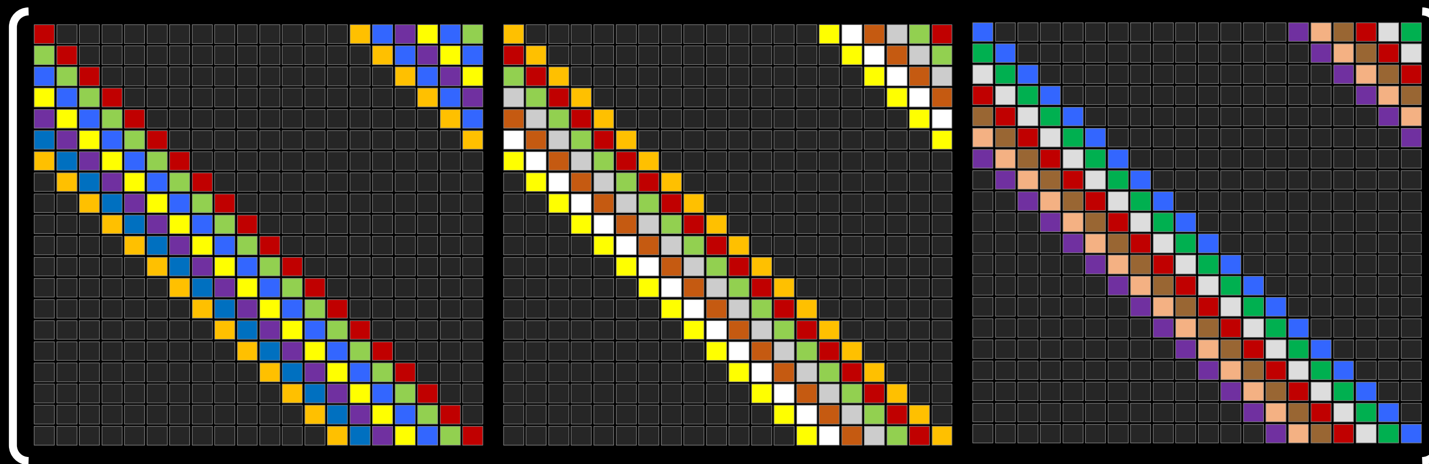
- $\mathbf{C}^{\mathrm{i}} \in \mathbb{R}^{N \times N}$ is a banded and Circulant matrix containing a single atom with all of its shifts



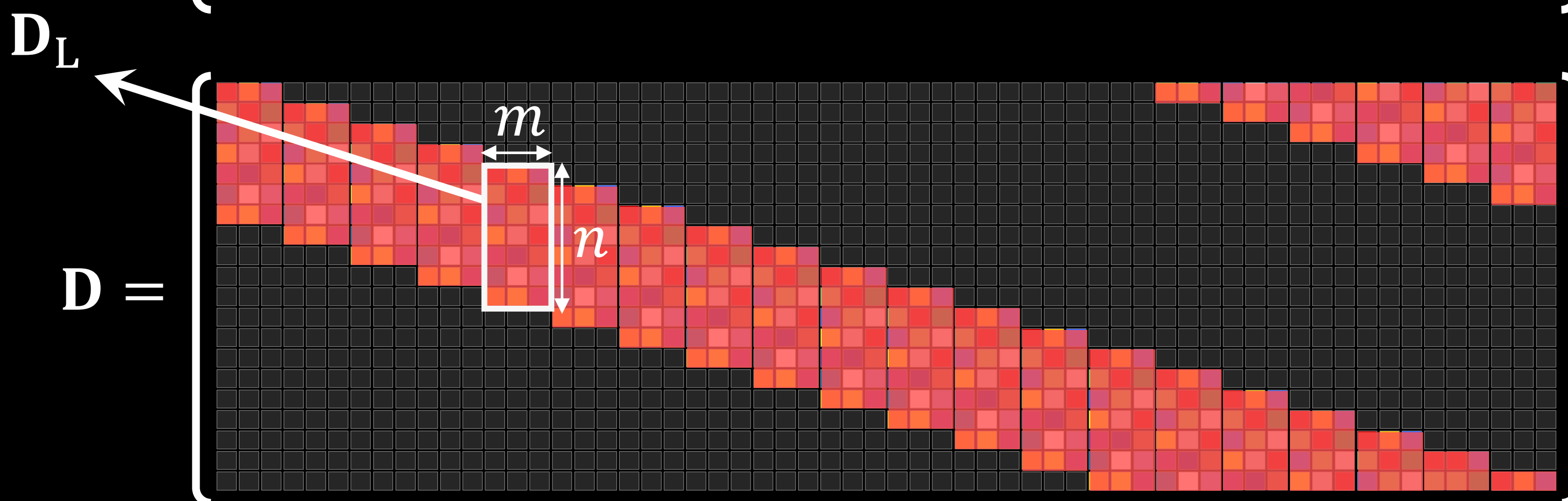

- $\mathbf{\Gamma}^{\mathrm{i}} \in \mathbb{R}^{N}$ are the corresponding coefficients ordered as column vectors

# The CSC Dictionary

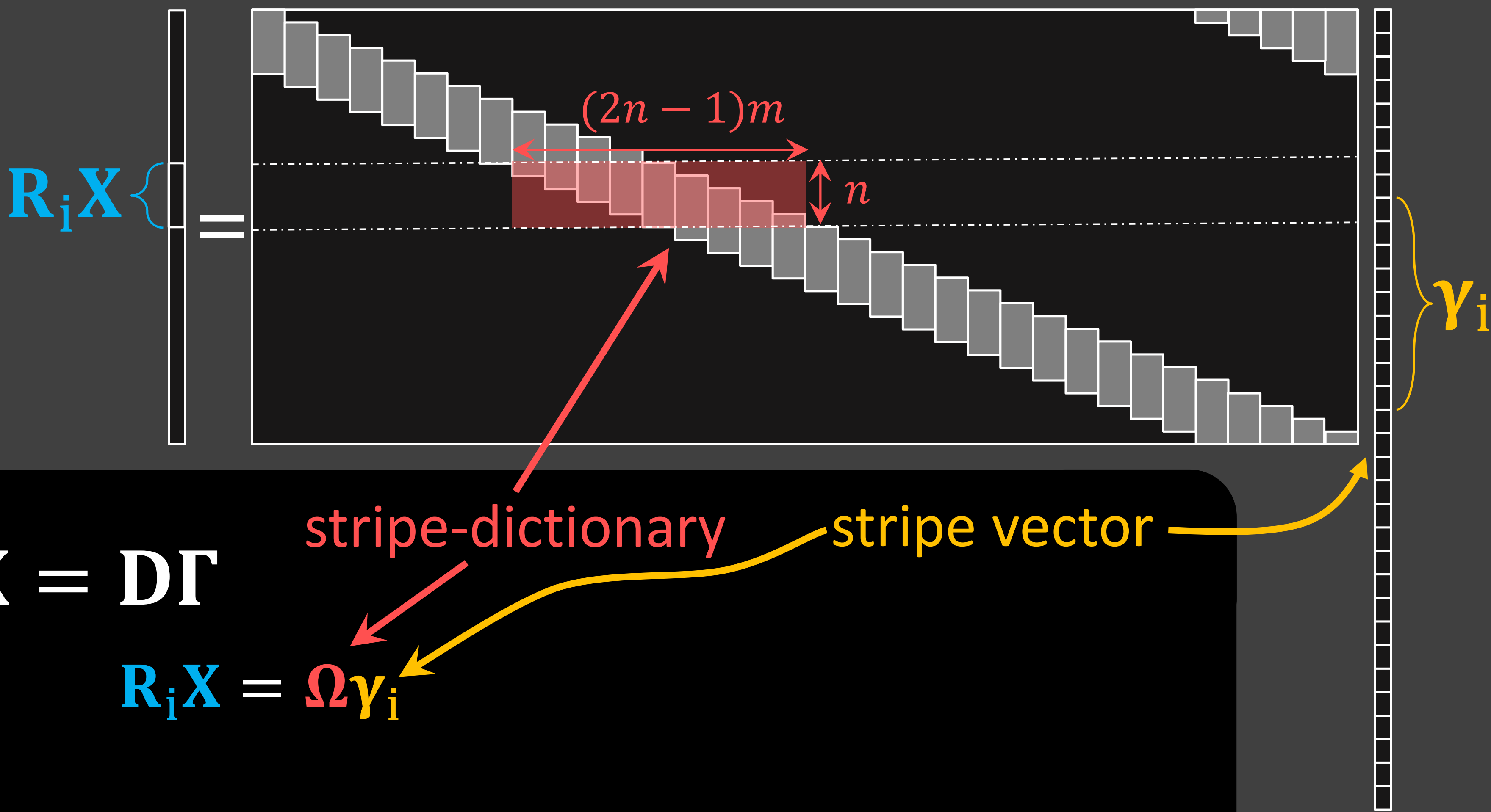
$[\mathbf{C}^1\ \mathbf{C}^2\ \mathbf{C}^3] =$



$\mathbf{D} =$



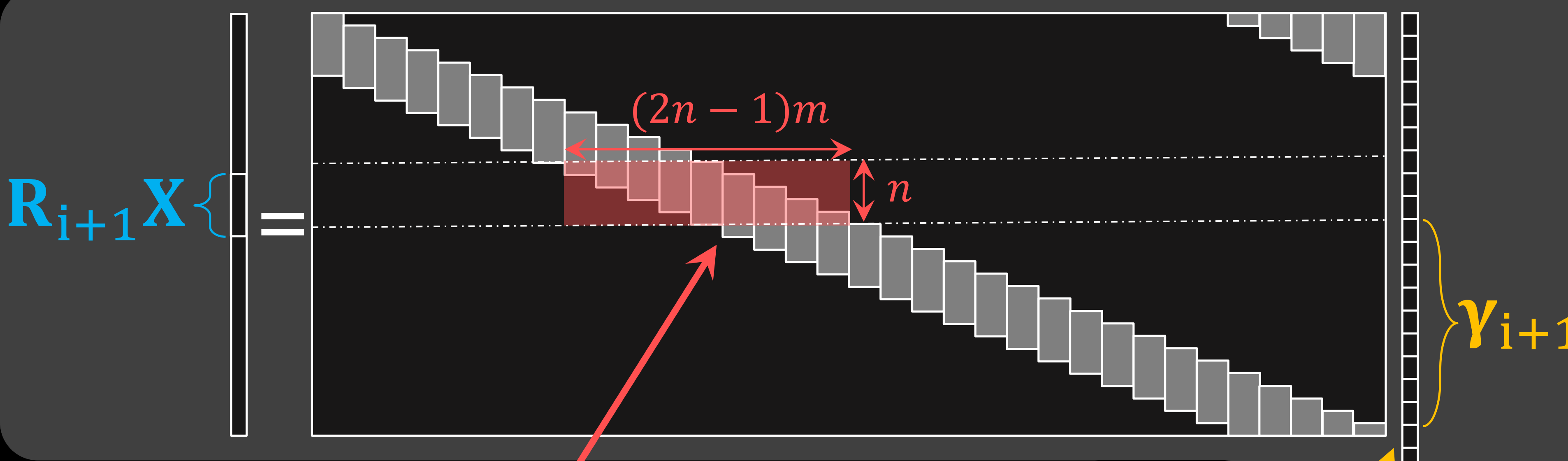
# Why CSC?




$$\mathbf{X} = \mathbf{D}\boldsymbol{\Gamma}$$

$$\mathbf{R}_i\mathbf{X} = \boldsymbol{\Omega}\boldsymbol{\gamma}_i$$

# Why CSC?




stripe-dictionary

stripe vector

$$\mathbf{X} = \mathbf{D}\boldsymbol{\Gamma}$$

$$\mathbf{R}_i\mathbf{X} = \boldsymbol{\Omega}\boldsymbol{\gamma}_i$$

$$\mathbf{R}_{i+1}\mathbf{X} = \boldsymbol{\Omega}\boldsymbol{\gamma}_{i+1}$$

Every patch has a sparse representation w.r.t. to the same local dictionary ($\boldsymbol{\Omega}$) just as assumed for images

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# Classical Sparse Theory for CSC ?

$$\min_{\mathbf{\Gamma}} \ \|\mathbf{\Gamma}\|_0 \ \ \text{s.t.} \ \|\mathbf{Y} - \mathbf{D}\mathbf{\Gamma}\|_2 \leq \varepsilon$$

**Theorem**: **BP is guaranteed to "succeed" …. if** $\|\mathbf{\Gamma}\|_0 < \frac{1}{4}\left(1 + \frac{1}{\mu}\right)$

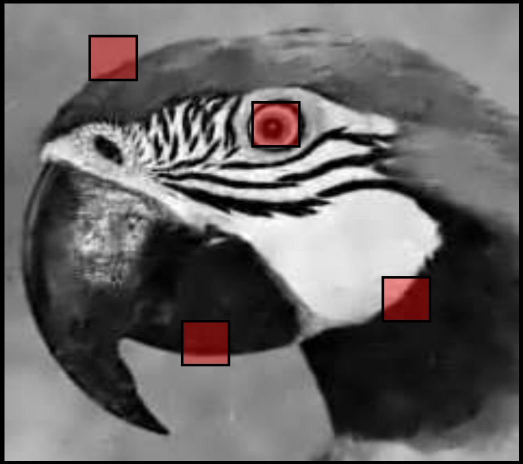
- Assuming that $m = 2$ and $n = 64$ we have that [Welch, '74]

$$\mu \geq 0.063$$

- Success of pursuits is guaranteed as long as

$$\|\mathbf{\Gamma}\|_0 < \frac{1}{4}\left(1 + \frac{1}{\mu(\mathbf{D})}\right) \leq \frac{1}{2}\left(1 + \frac{1}{0.063}\right) \approx 4.2$$

- Only few (4) non-zeros GLOBALLY are allowed!!! This is a very pessimistic result!



# Classical Sparse Theory for CSC ?

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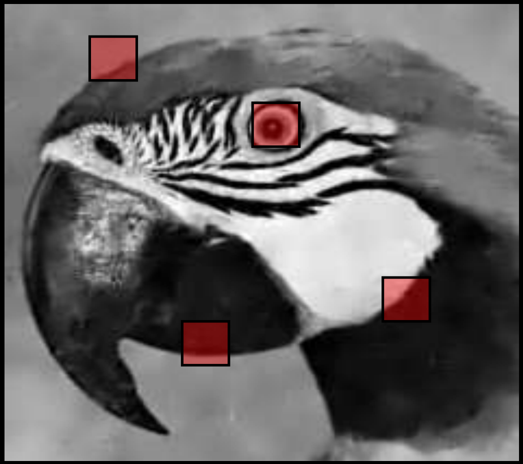
$$\mu \geq 0.063$$

- Success of pursuits i…

  …2

- O… …zeros GLOBALLY are all…ed!!! This is a very pessimistic result!

The classic *Sparseland* Theory does not provide good explanations for the CSC model



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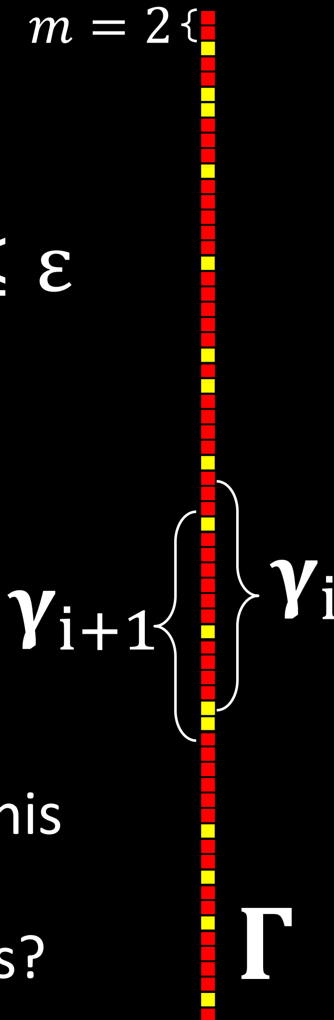
# Moving to Local Sparsity: Stripes

$\ell_{0,\infty}$ Norm: $\|\mathbf{\Gamma}\|_{0,\infty}^{s} = \max_{i} \|\boldsymbol{\gamma}_i\|_0$

$$\min_{\mathbf{\Gamma}} \ \|\mathbf{\Gamma}\|_{0,\infty}^{s} \ \text{s.t.} \ \|\mathbf{Y} - \mathbf{D}\mathbf{\Gamma}\|_2 \leq \varepsilon$$

$\|\mathbf{\Gamma}\|_{0,\infty}^{s}$ is low $\rightarrow$ all $\boldsymbol{\gamma}_i$ are sparse $\rightarrow$ every patch has a sparse representation over $\mathbf{\Omega}$




**The main question we aim to address is this:**

Can we generalize the vast theory of *Sparseland* to this new notion of local sparsity? For example, could we provide guarantees for success for pursuit algorithms?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# Success of the Basis Pursuit

$$\Gamma_{BP} = \min_{\Gamma} \frac{1}{2}\|Y - D\Gamma\|_2^2 + \lambda\|\Gamma\|_1$$

**Theorem:** For $Y = D\Gamma + E$, if $\lambda = 4\|E\|_{2,\infty}^{p}$, **if**

$$\|\Gamma\|_{0,\infty}^{s} < \frac{1}{3}\left(1 + \frac{1}{\mu(D)}\right)$$

**then Basis Pursuit performs very-well:**

1. The support of $\Gamma_{BP}$ is contained in that of $\Gamma$
2. $\|\Gamma_{BP} - \Gamma\|_\infty \leq 7.5\|E\|_{2,\infty}^{p}$
3. Every entry greater than $7.5\|E\|_{2,\infty}^{p}$ is found
4. $\Gamma_{BP}$ is unique

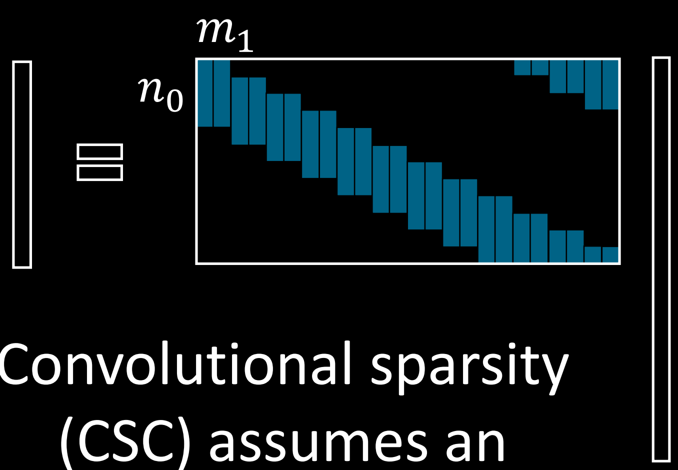
This is a much better result – it allows few non-zeros locally in each stripe, implying a permitted $O(N)$ non-zeros globally

Papyan, Sulam & Elad ('17)

# Multi-Layered Convolutional Sparse Modeling

# From CSC to Multi-Layered CSC

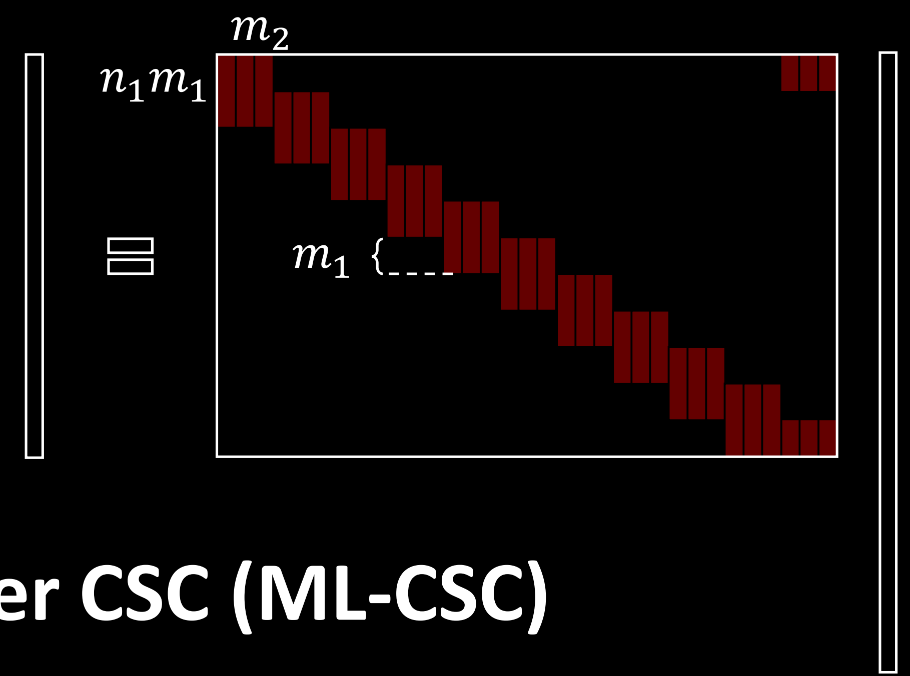
$\mathbf{X} \in \mathbb{R}^{N}$ $\mathbf{D}_1 \in \mathbb{R}^{N \times N m_1}$ $\boldsymbol{\Gamma}_1 \in \mathbb{R}^{N m_1}$




Convolutional sparsity (CSC) assumes an inherent structure is present in natural signals

We propose to impose the same structure on the representations themselves

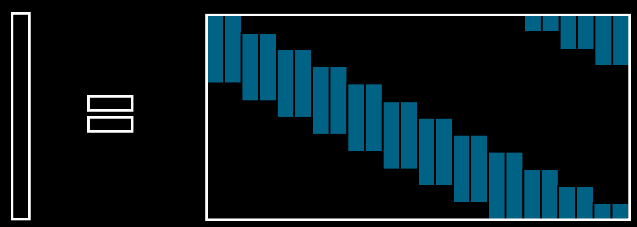
$\boldsymbol{\Gamma}_1 \in \mathbb{R}^{N m_1}$ $\mathbf{D}_2 \in \mathbb{R}^{N m_1 \times N m_2}$ $\boldsymbol{\Gamma}_2 \in \mathbb{R}^{N m_2}$

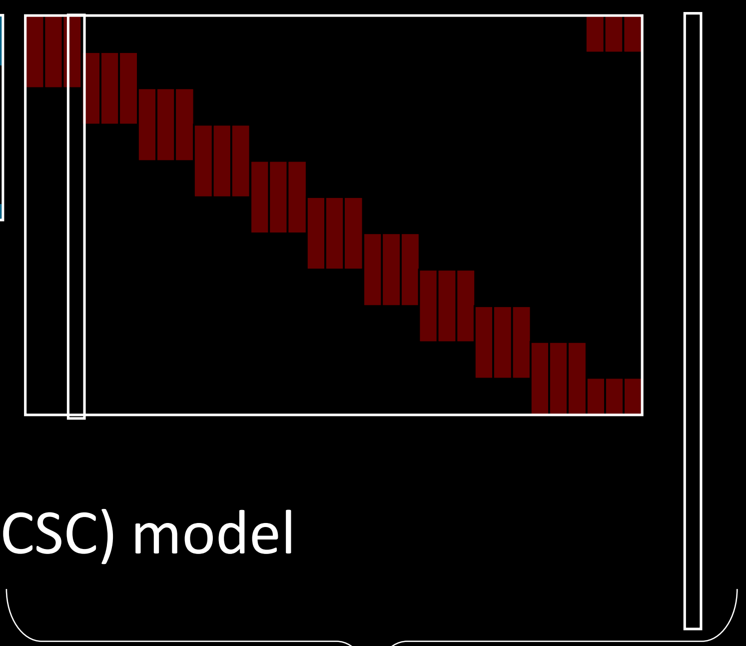



**Multi-Layer CSC (ML-CSC)**

# Intuition: From Atoms to Molecules

$\mathbf{X} \in \mathbb{R}^{N}$ $\quad \mathbf{D}_1 \in \mathbb{R}^{N\times Nm_1}$ $\quad \mathbf{\Gamma}_1 \in \mathbb{R}^{Nm_1}$ $\quad \mathbf{D}_2 \in \mathbb{R}^{Nm_1\times Nm_2}$ $\quad \mathbf{\Gamma}_2 \in \mathbb{R}^{Nm_2}$





- We can chain the all the dictionaries into one effective dictionary $\mathbf{D}_{\text{eff}} = \mathbf{D}_1\mathbf{D}_2\mathbf{D}_3\cdots\mathbf{D}_{\text{K}} \;\rightarrow\; \mathbf{x} = \mathbf{D}_{\text{eff}}\,\mathbf{\Gamma}_{\text{K}}$
- This is a special *Sparseland* (indeed, a CSC) model

$\mathbf{\Gamma}_1 \in \mathbb{R}^{Nm_1}$

- However:
  - A key property in this model: sparsity of the intermediate representations
  - The effective atoms: atoms

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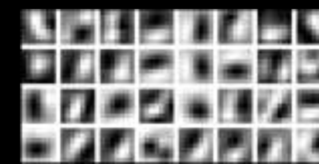
# A Small Taste: Model Training (MNIST)

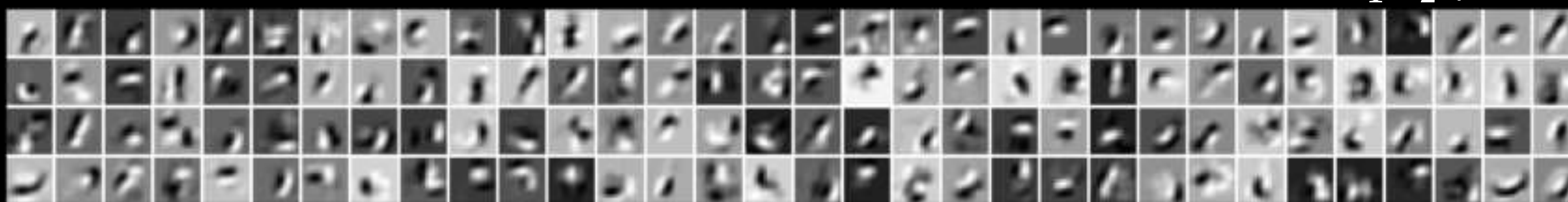
MNIST Dictionary:

- $D_1$: 32 filters of size 7×7, with stride of 2 (dense)
- $D_2$: 128 filters of size 5×5×32 with stride of 1 - 99.09 % sparse
- D3: 1024 filters of size 7×7×128 – 99.89 % sparse

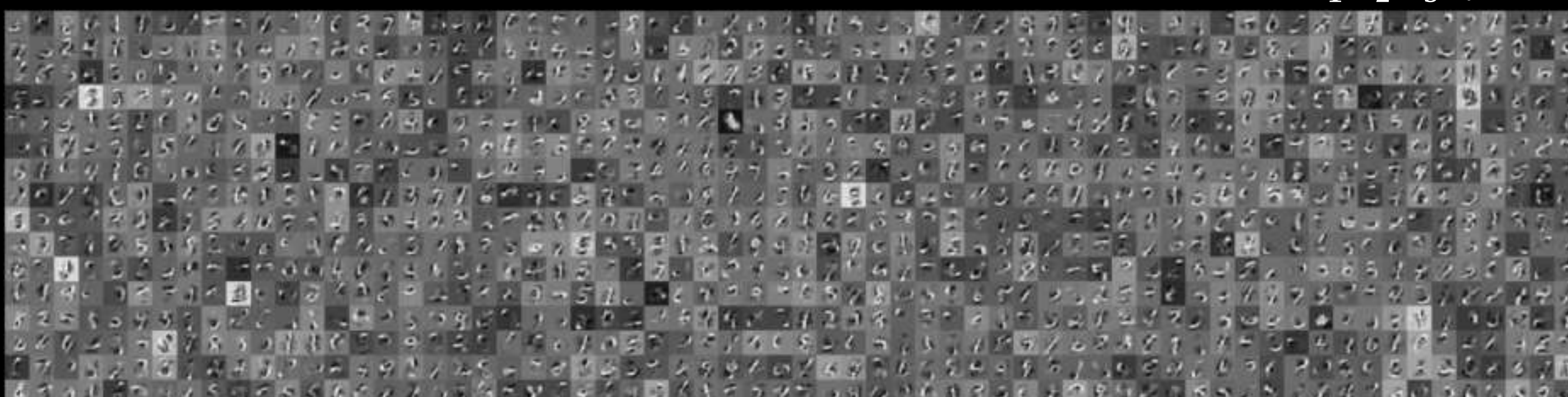
$\mathbf{D}_1$ (7×7)



$\mathbf{D}_1\mathbf{D}_2$ (15×15)



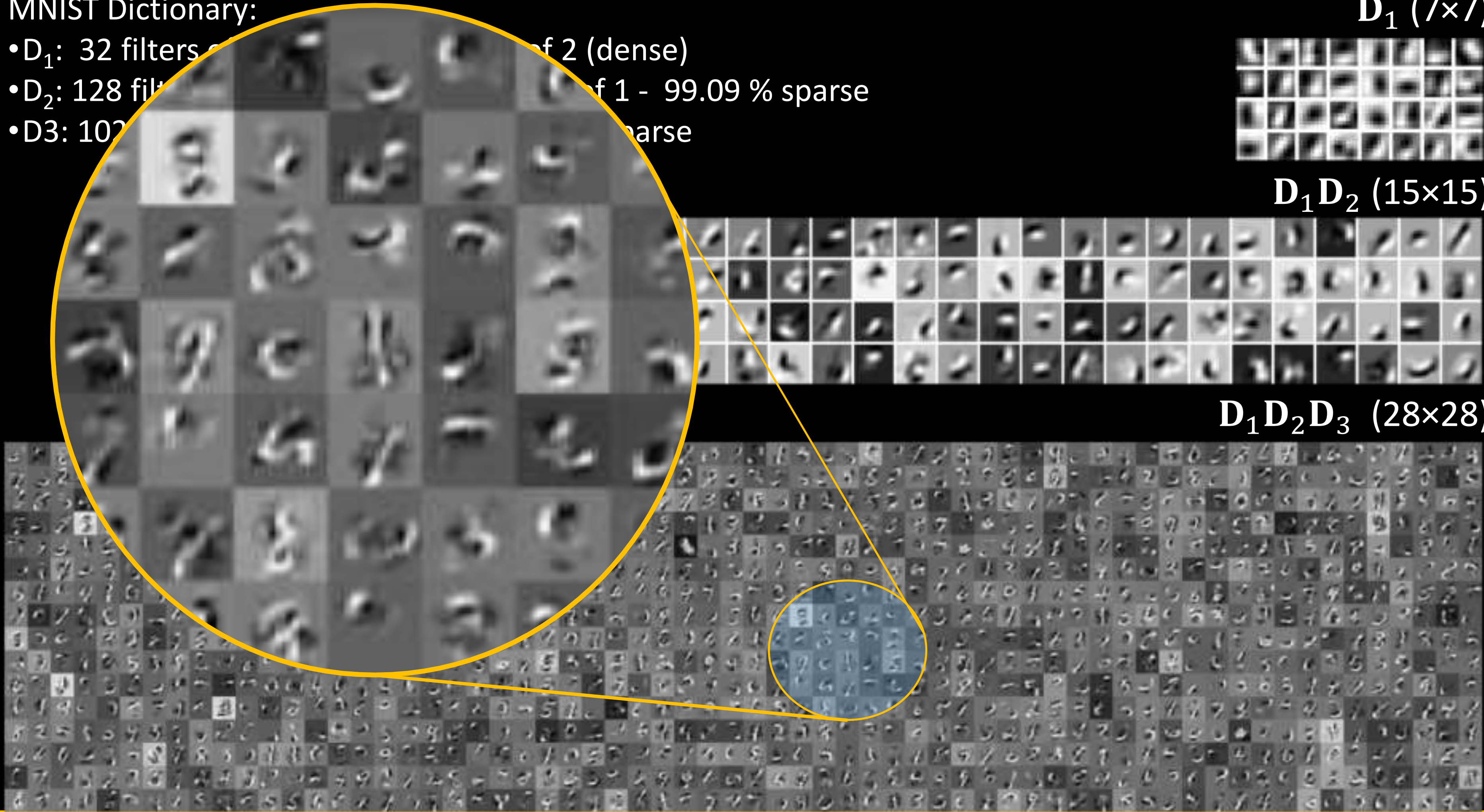
$\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3$ (28×28)



# A Small Taste: Model Training (MNIST)

MNIST Dictionary:

- $D_1$: 32 filters … f 2 (dense)
- $D_2$: 128 fil… f 1 - 99.09 % sparse
- D3: 102… arse



# ML-CSC: Pursuit

- Deep–Coding Problem $(\mathbf{DCP}_\lambda)$ (dictionaries are known):

$$\begin{Bmatrix} \mathbf{X} = \mathbf{D}_1\mathbf{\Gamma}_1 & \|\mathbf{\Gamma}_1\|_{0,\infty}^{s} \le \lambda_1 \\ \mathbf{\Gamma}_1 = \mathbf{D}_2\mathbf{\Gamma}_2 & \|\mathbf{\Gamma}_2\|_{0,\infty}^{s} \le \lambda_2 \\ \vdots & \vdots \\ \mathbf{\Gamma}_{K-1} = \mathbf{D}_K\mathbf{\Gamma}_K & \|\mathbf{\Gamma}_K\|_{0,\infty}^{s} \le \lambda_K \end{Bmatrix}$$

- Or, more realistically for noisy signals,

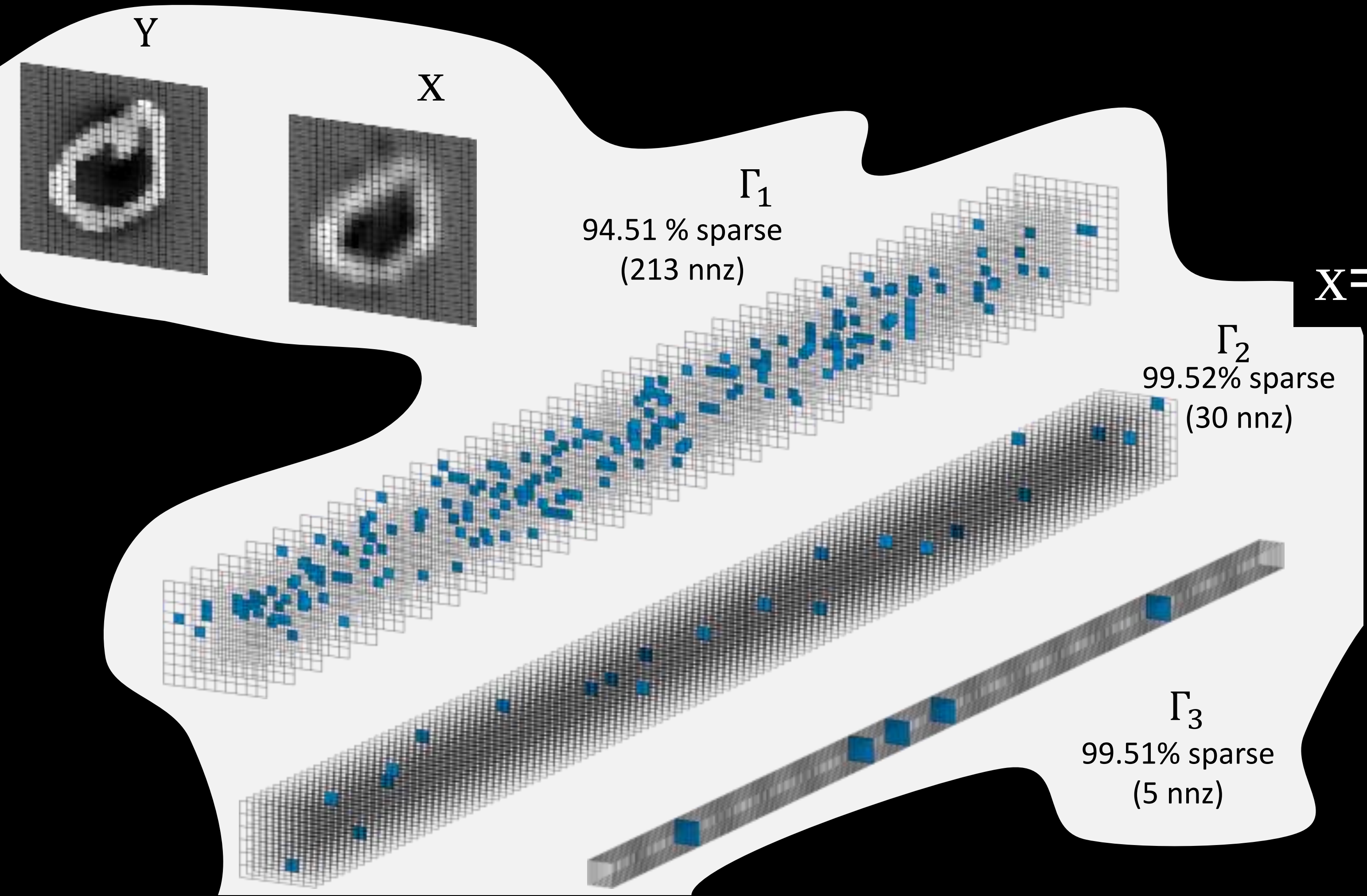
$$\text{Find } \{\mathbf{\Gamma}_j\}_{j=1}^{K} \quad s.t. \quad \begin{Bmatrix} \|\mathbf{Y} - \mathbf{D}_1\mathbf{\Gamma}_1\|_2 \le \mathcal{E} & \|\mathbf{\Gamma}_1\|_{0,\infty}^{s} \le \lambda_1 \\ \mathbf{\Gamma}_1 = \mathbf{D}_2\mathbf{\Gamma}_2 & \|\mathbf{\Gamma}_2\|_{0,\infty}^{s} \le \lambda_2 \\ \vdots & \vdots \\ \mathbf{\Gamma}_{K-1} = \mathbf{D}_K\mathbf{\Gamma}_K & \|\mathbf{\Gamma}_K\|_{0,\infty}^{s} \le \lambda_K \end{Bmatrix}$$

# A Small Taste: Pursuit

$x = \mathbf{D}_1 \Gamma_1$

$x = \mathbf{D}_1 \mathbf{D}_2 \Gamma_2$

$x = \mathbf{D}_1 \mathbf{D}_2 \mathbf{D}_3 \Gamma_3$

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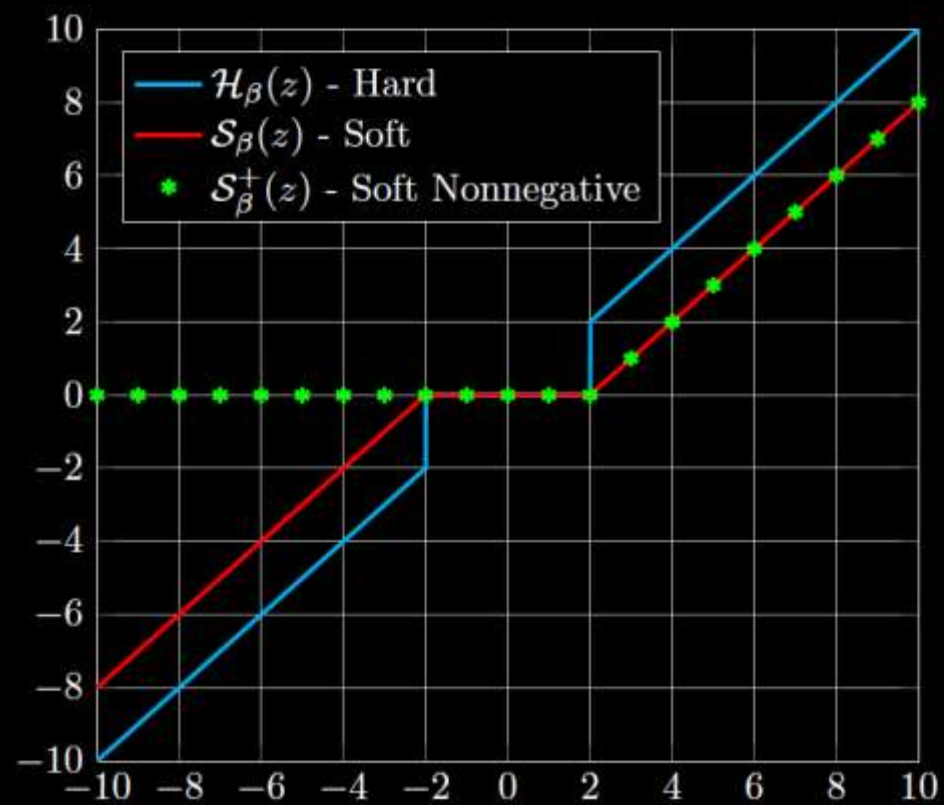
# ML-CSC: The Simplest Pursuit


The simplest pursuit algorithm (single-layer case) is the THR algorithm, which operates on a given input signal $\mathbf{Y}$ by:

$$\mathbf{Y} = \mathbf{D}\mathbf{\Gamma} + \mathbf{E}$$

and $\mathbf{\Gamma}$ is sparse

$$\hat{\mathbf{\Gamma}} = \mathcal{P}_\beta\left(\mathbf{D}^{\mathrm{T}}\mathbf{Y}\right)$$

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# Consider this for Solving the DCP

- Layered thresholding (LT):

Estimate $\mathbf{\Gamma}_1$ via the THR algorithm

$$\hat{\mathbf{\Gamma}}_2 = \mathcal{P}_{\beta_2}\left(\mathbf{D}_2^T \mathcal{P}_{\beta_1}\left(\mathbf{D}_1^T \mathbf{Y}\right)\right)$$

Estimate $\mathbf{\Gamma}_2$ via the THR algorithm

$$\left(\mathbf{DCP}_\lambda^\mathcal{E}\right): \text{ Find } \left\{\mathbf{\Gamma}_j\right\}_{j=1}^K \quad s.t.$$

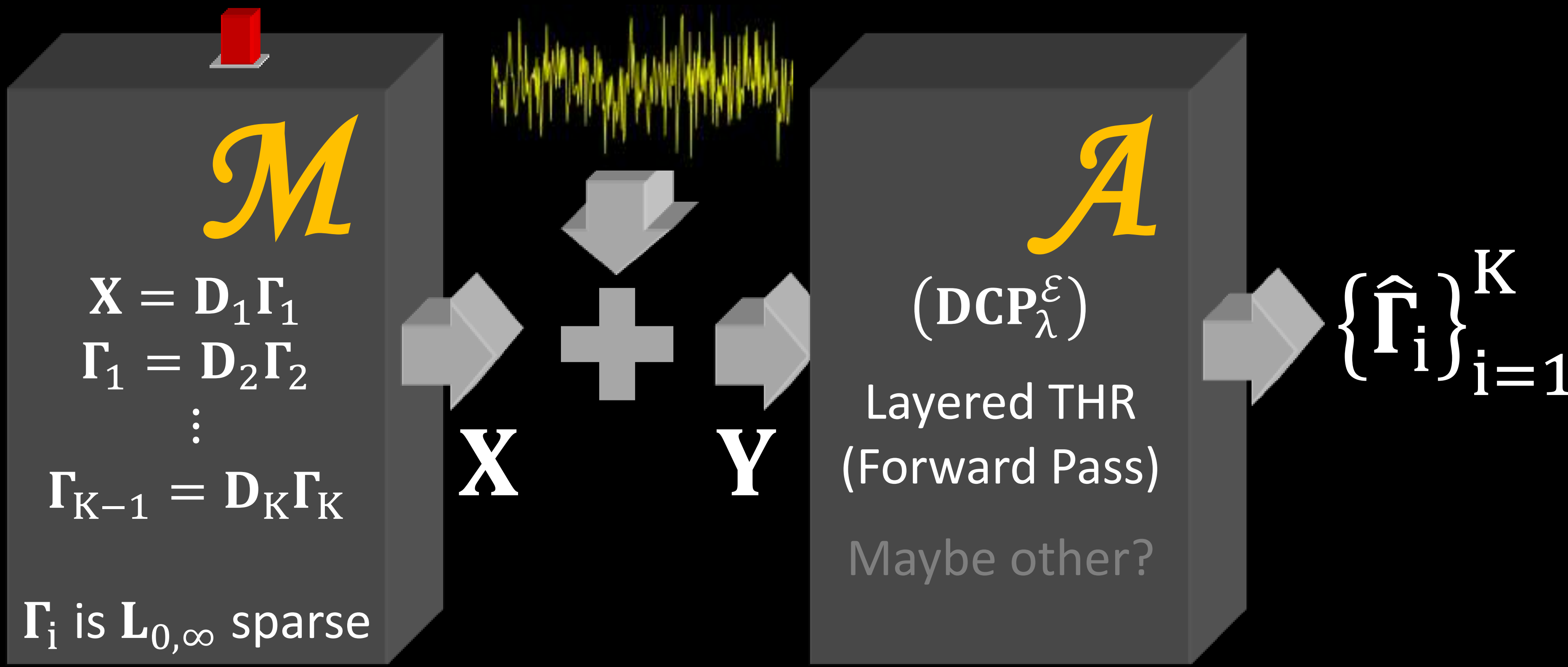
$$\left\{\begin{array}{ll} \|\mathbf{Y} - \mathbf{D}_1\mathbf{\Gamma}_1\|_2 \leq \mathcal{E} & \|\mathbf{\Gamma}_1\|_{0,\infty}^s \leq \lambda_1 \\ \mathbf{\Gamma}_1 = \mathbf{D}_2\mathbf{\Gamma}_2 & \|\mathbf{\Gamma}_2\|_{0,\infty}^s \leq \lambda_2 \\ \vdots & \vdots \\ \mathbf{\Gamma}_{K-1} = \mathbf{D}_K\mathbf{\Gamma}_K & \|\mathbf{\Gamma}_K\|_{0,\infty}^s \leq \lambda_K \end{array}\right\}$$

- Now let’s take a look at how Conv. Neural Network operates:

$$f(\mathbf{Y}) = \text{ReLU}\left(\mathbf{b}_2 + \mathbf{W}_2^T \text{ ReLU}\left(\mathbf{b}_1 + \mathbf{W}_1^T \mathbf{Y}\right)\right)$$

The layered (soft nonnegative) thresholding and the CNN forward pass algorithm are the very same thing !!!

# Theoretical Path




Armed with this view of a generative source model, we may ask new and daring theoretical questions

# Success of the Layered-THR

**Theorem:** If $\|\boldsymbol{\Gamma}_i\|_{0,\infty}^{s} < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{D}_i)} \cdot \frac{|\Gamma_i^{min}|}{|\Gamma_i^{max}|}\right) - \frac{1}{\mu(\mathbf{D}_i)} \cdot \frac{\varepsilon_L^{i-1}}{|\Gamma_i^{max}|}$

then the **Layered Hard THR** (with the proper thresholds) **finds the correct supports** and $\left\|\boldsymbol{\Gamma}_i^{LT} - \boldsymbol{\Gamma}_i\right\|_{2,\infty}^{p} \leq \varepsilon_L^i$, where we have defined $\varepsilon_L^0 = \|\mathbf{E}\|_{2,\infty}^{p}$ and

$$\varepsilon_L^i = \sqrt{\|\boldsymbol{\Gamma}_i\|_{0,\infty}^{p}} \cdot \left(\varepsilon_L^{i-1} + \mu(\mathbf{D}_i)\left(\|\boldsymbol{\Gamma}_i\|_{0,\infty}^{s} - 1\right)|\Gamma_i^{max}|\right)$$

Papyan, Romano & Elad ('17)

The stability of the forward pass is guaranteed if the underlying representations are **locally** sparse and the noise is **locally** bounded

**Problems:**

1. Contrast
2. Error growth
3. Error even if no noise

# Layered Basis Pursuit (BP)

- We chose the Thresholding algorithm due to its simplicity, but we do know that there are better pursuit methods – how about using them?
- Lets use the Basis Pursuit instead …

$$\left(\mathbf{DCP}_{\lambda}^{\mathcal{E}}\right): \text{ Find } \left\{\boldsymbol{\Gamma}_{\mathrm{j}}\right\}_{\mathrm{j}=1}^{\mathrm{K}} \quad s.t.$$

$$\left(\begin{array}{ll} \|\mathbf{Y}-\mathbf{D}_1\boldsymbol{\Gamma}_1\|_2 \leq \mathcal{E} & \|\boldsymbol{\Gamma}_1\|_{0,\infty}^{\mathrm{s}} \leq \lambda_1 \\ \boldsymbol{\Gamma}_1 = \mathbf{D}_2\boldsymbol{\Gamma}_2 & \|\boldsymbol{\Gamma}_2\|_{0,\infty}^{\mathrm{s}} \leq \lambda_2 \\ \vdots & \vdots \\ \boldsymbol{\Gamma}_{\mathrm{K}-1} = \mathbf{D}_{\mathrm{K}}\boldsymbol{\Gamma}_{\mathrm{K}} & \|\boldsymbol{\Gamma}_{\mathrm{K}}\|_{0,\infty}^{\mathrm{s}} \leq \lambda_{\mathrm{K}} \end{array}\right)$$

$$\boldsymbol{\Gamma}_1^{\mathrm{LBP}} = \min_{\boldsymbol{\Gamma}_1} \frac{1}{2}\|\mathbf{Y}-\mathbf{D}_1\boldsymbol{\Gamma}_1\|_2^2 + \lambda_1\|\boldsymbol{\Gamma}_1\|_1$$

$$\Downarrow$$

$$\boldsymbol{\Gamma}_2^{\mathrm{LBP}} = \min_{\boldsymbol{\Gamma}_2} \frac{1}{2}\left\|\boldsymbol{\Gamma}_1^{\mathrm{LBP}}-\mathbf{D}_2\boldsymbol{\Gamma}_2\right\|_2^2 + \lambda_2\|\boldsymbol{\Gamma}_2\|_1$$

$$\Downarrow$$

$$\vdots$$

**Deconvolutional networks**
[Zeiler, Krishnan, Taylor & Fergus '10]

# Success of the Layered BP

**Theorem: Assuming that** $\|\mathbf{\Gamma_i}\|_{0,\infty}^{s} < \frac{1}{3}\left(1 + \frac{1}{\mu(\mathbf{D_i})}\right)$
then the Layered Basis Pursuit performs very well:

1. The support of $\mathbf{\Gamma}_i^{LBP}$ is contained in that of $\mathbf{\Gamma}_i$
2. The error is bounded: $\|\mathbf{\Gamma}_i^{LBP} - \mathbf{\Gamma}_i\|_{2,\infty}^{p} \leq \varepsilon_L^i$, where
$$\varepsilon_L^i = 7.5^i \|\mathbf{E}\|_{2,\infty}^{p} \prod_{j=1}^{i} \sqrt{\|\mathbf{\Gamma}_j\|_{0,\infty}^{p}}$$
3. Every entry in $\mathbf{\Gamma}_i$ greater than $\varepsilon_L^i / \sqrt{\|\mathbf{\Gamma}_i\|_{0,\infty}^{p}}$ will be found

Papyan, Romano & Elad ('17)

**Problems:**
1. ~~Contrast~~
2. Error growth
3. ~~Error even if no noise~~

# Layered Iterative Thresholding

Layered BP: $\Gamma_j^{LBP} = \min_{\Gamma_j} \frac{1}{2} \left\| \Gamma_{j-1}^{LBP} - \mathbf{D}_j \Gamma_j \right\|_2^2 + \xi_j \left\| \Gamma_j \right\|_1$ j

Layered Iterative Soft-Thresholding:

t $\Gamma_j^t = \mathcal{S}_{\xi_j/c_j}\left(\Gamma_j^{t-1} + \mathbf{D}_j^T\left(\hat{\Gamma}_{j-1} - \mathbf{D}_j\Gamma_j^{t-1}\right)\right)$ j

Note that our suggestion implies that groups of layers share the same dictionaries

Can be seen as a very deep recurrent neural network

[Gregor & LeCun '10]

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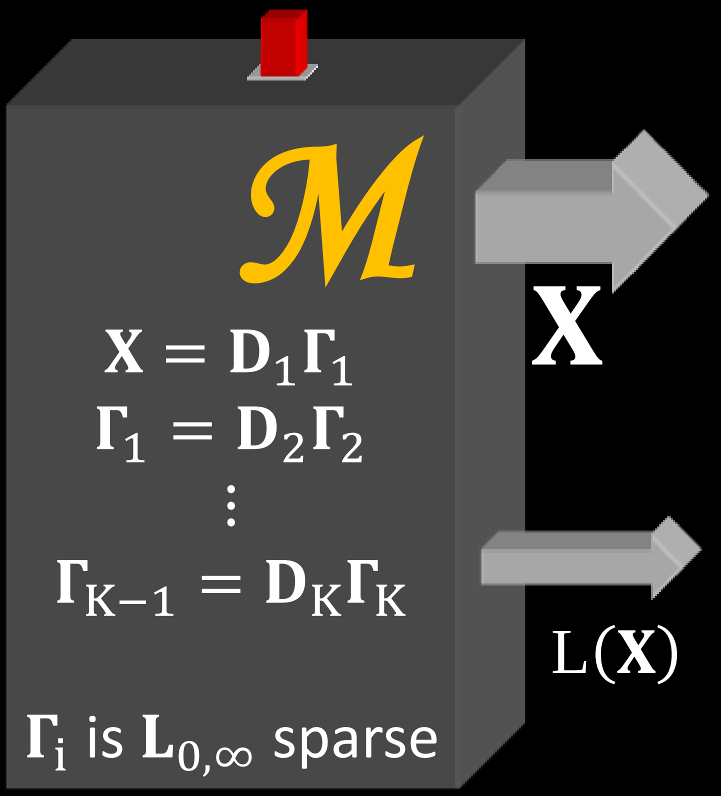
# Where are the Labels?




We presented the ML-CSC as a machine that produces signals $\mathbf{X}$

Answer 1:

- We do not need labels because everything we show refer to the unsupervised case, in which we operate on signals, not necessarily in the context of recognition

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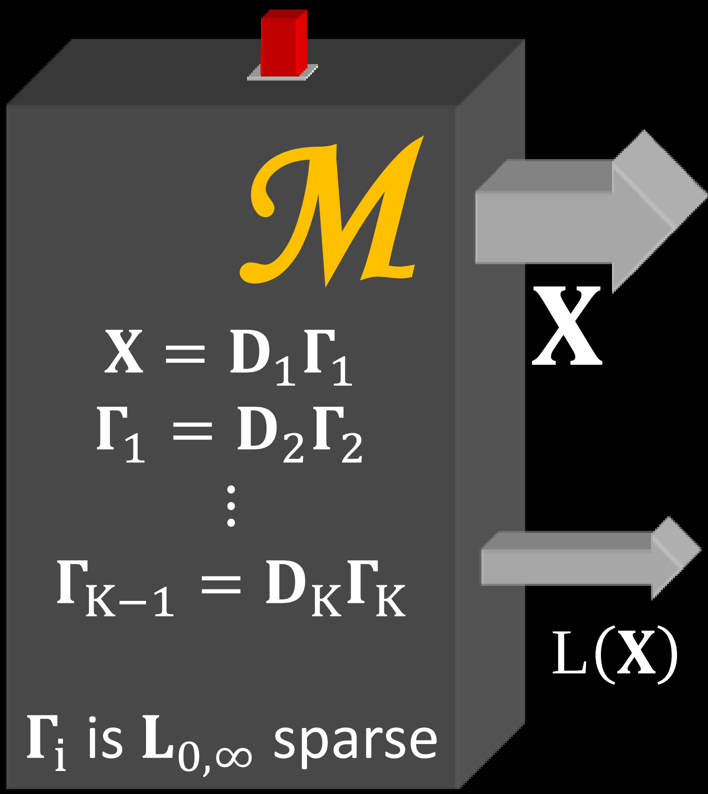
# Where are the Labels?




We presented the ML-CSC as a machine that produces signals $\mathbf{X}$

Answer 2:

- In fact, this model could be augmented by a synthesis of the corresponding label by:

$$\mathrm{L}(\mathbf{X}) = sign\left\{c + \sum_{j=1}^{K} w_j^T \Gamma_j\right\}$$

- This assumes that knowing the representations (or maybe their supports?) suffice for identifying the label
- Thus, a successful pursuit algorithm can lead to an accurate recognition if the network is augmented by a FC classification layer

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# What About Learning?

**Sparseland**
Sparse Representation Theory

**CSC**
Convolutional Sparse Coding

**ML-CSC**
Multi-Layered Convolutional Sparse Coding

All these models rely on proper
Dictionary Learning Algorithms to fulfil their mission:

- Sparseland: We have unsupervised and supervised such algorithms, and a beginning of theory to explain how these work
- CSC: We have few and only unsupervised methods, and even these are not fully stable/clear
- ML-CSC: One algorithm has been proposed (unsupervised) – see ArxiV

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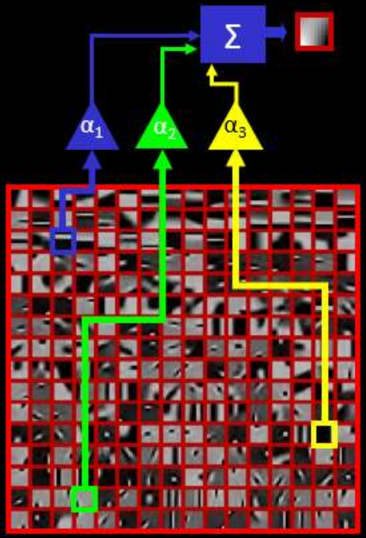
# Time to Conclude

# This Talk

*Sparseland*

The desire to model data



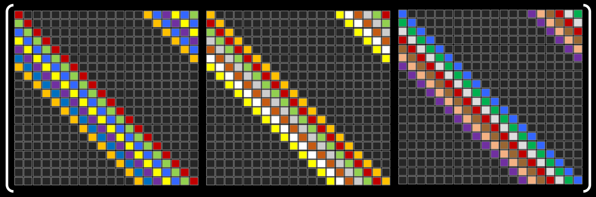

We spoke about the importance of models in signal/image processing and described *Sparseland* in details

# This Talk

*Sparseland*

The desire to model data

Novel View of Convolutional Sparse Coding



We presented a theoretical study of the CSC model and how to operate locally while getting global optimality

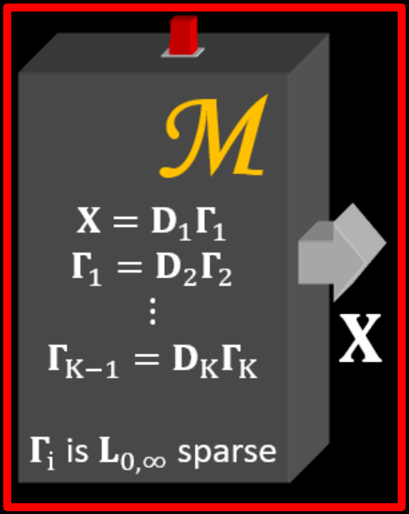
# This Talk

*Sparseland*

The desire to model data

Novel View of Convolutional Sparse Coding

Multi-Layer Convolutional Sparse Coding



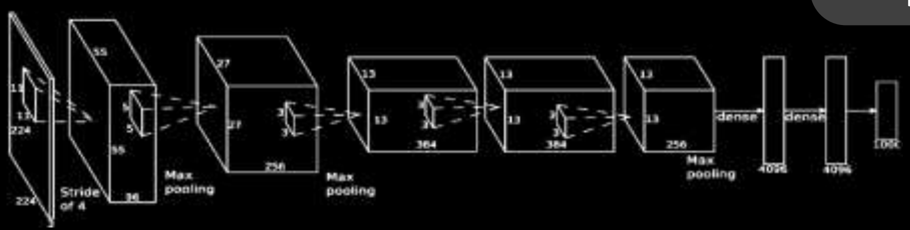

We propose a multi-layer extension of CSC, shown to be tightly connected to CNN

# This Talk

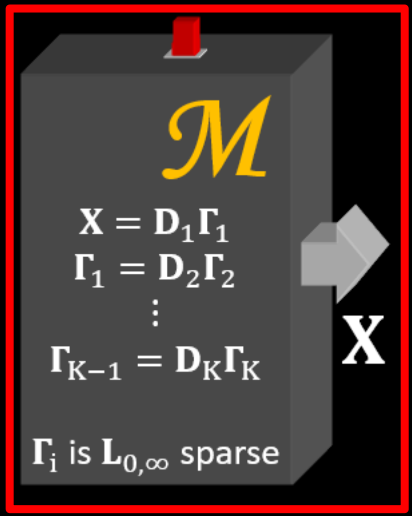
*Sparseland*

The desire to model data

Novel View of Convolutional Sparse Coding



Multi-Layer Convolutional Sparse Coding




A novel interpretation and theoretical understanding of CNN

The ML-CSC was shown to enable a theoretical study of CNN, along with new insights

# This Talk

*Sparseland*

The desire to model data

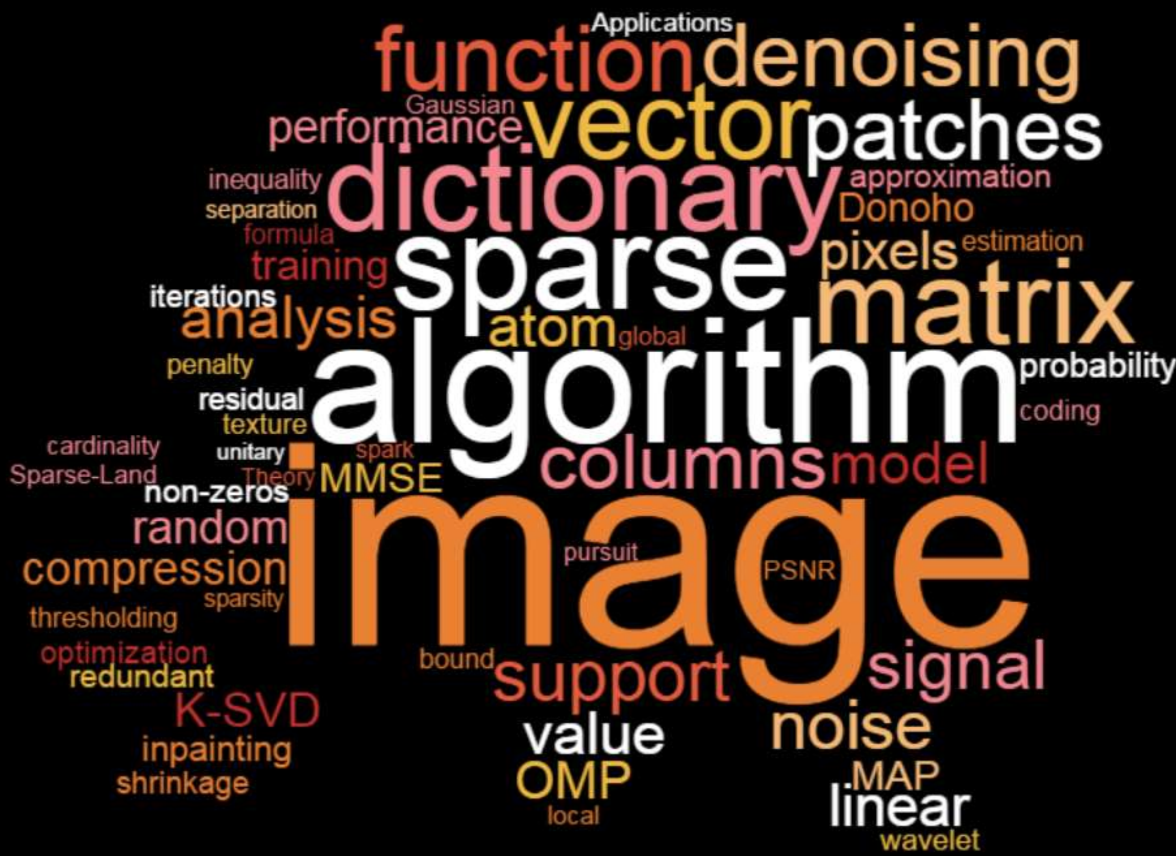
Novel View of Convolutional Sparse Coding

Multi-Layer Convolutional Sparse Coding

A novel interpretation and theoretical understanding of CNN

**Take Home Message 1:**
Generative modeling of data sources enables algorithm development along with theoretically analyzing algorithms' performance

**Take Home Message 2:**
The Multi-Layer Convolutional Sparse Coding model could be a new platform for understanding and developing deep-learning solutions




More on these (including these slides and the relevant papers) can be found in http://www.cs.technion.ac.il/~elad

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