MMSE Approximation for the Sparse Prior

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Joint work with Jeremias Sulam, Yaniv Romano, Yue M. Lu and Michael Elad
Noise Removal

Why denoising?
A simple testing ground for novel concepts in signal processing.
Can be generalized to other, more complicated applications.
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Dror Simon (Technion)
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Noise Removal
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Noisy Signal → Signal Restoration → Clean Estimate
Noise Removal

Noisy Signal

Signal Restoration

Clean Estimate

Prior Knowledge
Denoising
Noise Removal – Bayesian Standpoint

Denoising \rightarrow Signal Estimation \rightarrow Approx.
Noise Removal – Bayesian Standpoint

Denoising → Suboptimal Signal Estimation → Approx.
Noise Removal – Bayesian Standpoint

- Denoising
- Suboptimal Signal Estimation
- Optimal Signal Estimation
- Approx.
Noise Removal – Bayesian Standpoint

Denoising

Suboptimal Signal Estimation

Approx.

Optimal Signal Estimation

Approx.
1. Bayesian Framework
   - The Generative Model
   - Bayesian Estimators

2. MMSE Approximation
   - Previous Work

3. Stochastic Resonance
   - Can Noise Help Denoising?

4. Our Proposed Method
   - The Algorithm
   - Unitary Case Analysis
   - The General Dictionary Case
   - Image Denoising

5. Conclusions
Outline

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5. Conclusions
$D \in \mathbb{R}^{n \times m}$ is a dictionary with normalized columns.
Each element $i$ in $\alpha$ is non zero with probability $p_i \ll 1$. 
The non-zero elements of the sparse representation, denoted by $\alpha_s$, are sampled from a Gaussian distribution $\alpha_s|s \sim \mathcal{N}(0, \sigma^2_{\alpha |s|})$. 
The product $D\alpha$ leads to a signal $x$. 

$$D\alpha = x$$
The Generative Model

We are given noisy measurements $y = D\alpha + \nu$, where $\nu$ is a white Gaussian noise $\nu \sim \mathcal{N}(0, \sigma^2\nu I_n)$.
The Generative Model – Results

The prior probability of a support (Bernoulli):

\[ p(s) = \prod_{i \in s} p_i \prod_{j / \in s} (1 - p_j). \]

When the support is known, \( y \) and \( \alpha_s \) are jointly Gaussian:

\[ y = D_s \alpha_s + \nu, \]

leading to:

\[ y \mid s \text{ is Gaussian:} \quad y \mid s \sim \mathcal{N}(0, C_s). \]

\[ y \mid \alpha_s, s \text{ is Gaussian:} \quad y \mid \alpha_s, s \sim \mathcal{N}(D_s \alpha_s, \sigma^2 \nu I). \]

\[ \alpha_s \mid y, s \text{ is Gaussian:} \quad \alpha_s \mid y, s \sim \mathcal{N}(\frac{1}{\sigma^2 \nu} Q_s^{-1} D_s^T y, Q_s^{-1}). \]

\[ C_s = \sigma^2 \alpha_s D_s D_s^T + \sigma^2 \nu I, \]

\[ Q_s = \frac{1}{\sigma^2 \nu} I_{|s|} + \frac{1}{\sigma^2 \nu} D_s^T D_s. \]

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  \[ y = D_s \alpha_s + \nu, \text{ leading to} \]
  - \( y|s \) is Gaussian: \( y|s \sim \mathcal{N}(0, C_s) \).
  - \( y|\alpha_s, s \) is Gaussian: \( y|\alpha_s, s \sim \mathcal{N}(D_s \alpha_s, \sigma^2 \nu I_n) \).

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    - \( y|s \) is Gaussian: \( y|s \sim N(0, C_s) \).
    - \( y|\alpha_s, s \) is Gaussian: \( y|\alpha_s, s \sim N(D_s \alpha_s, \sigma^2 \nu \mathbf{I}_n) \).
    - \( \alpha_s|y, s \) is Gaussian: \( \alpha_s|y, s \sim N \left( \frac{1}{\sigma^2 \nu} Q_s^{-1} D_s^T y, Q_s^{-1} \right) \).

\[
C_s = \sigma^2 \alpha D_s D_s^T + \sigma^2 \nu \mathbf{I}_n, \quad Q_s = \frac{1}{\sigma^2 \alpha} |s| \mathbf{I} + \frac{1}{\sigma^2 \nu} D_s^T D_s
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   - Bayesian Estimators

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**The goal:** Estimate $\alpha$ given the noisy measurements $y$, i.e. denoise the signal.
Bayesian Estimators

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Many estimators can be proposed. We focus our attention on three:
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2. The Maximum A-posteriori Probability (MAP) estimator.
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Many estimators can be proposed. We focus our attention on three:

1. The oracle estimator.
2. The Maximum A-posteriori Probability (MAP) estimator.
3. The Minimum Mean Square Error (MMSE) estimator.
Oracle Estimator

\[ \hat{\alpha}_s^{\text{Oracle}} = \mathbb{E} \{ \alpha_s \mid s, y \} = \frac{1}{\sigma^2_{\nu}} Q_s^{-1} D_s^T y \]
The Bayesian Estimators

MAP Support Estimator

\[ \hat{s}_{\text{MAP}} = \arg \max_s p(s | y) \]
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\[ = \arg \max_s -\frac{1}{2} y^T C_s^{-1} y - \frac{1}{2} \log \det(C_s) \]

\[ + \sum_{i \in s} \log(p_i) + \sum_{j \notin s} \log(1 - p_j) \]
The Bayesian Estimators

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where \( s \in \{0, 1\}^m \)
The Bayesian Estimators

**MMSE Estimator**

\[
\hat{\alpha}_{\text{MMSE}} = \arg\min_{\hat{\alpha}(y)} \mathbb{E} \left\{ \| \hat{\alpha}(y) - \alpha \|_2^2 \mid y \right\}
\]

MMSE for Sparse Prior

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The Bayesian Estimators

**MMSE Estimator**

\[ \hat{\alpha}_{\text{MMSE}} = \arg \min_{\hat{\alpha}(y)} \mathbb{E} \left\{ \| \hat{\alpha}(y) - \alpha \|_2^2 \left| y \right\} \right\} = \mathbb{E} \{ \alpha | y \} \]
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\[ = \mathbb{E}_{s \big| y} \left\{ \mathbb{E}_{\alpha \big| y, s} \left\{ \alpha \big| y, s \right\} \right\} \]
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Both estimators are practically impossible to obtain.
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- MAP – compute the posterior probability of each of the $2^m$ supports and pick the most probable one.
- MMSE – compute the posterior probability of each support and use them as weights for all the possible oracle estimators.

How is this issue resolved?
- MAP – use an approximation algorithm (greedy or relaxed) to recover a likely support, and then use the oracle.
- MMSE – usually avoided.
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   - The Generative Model
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2. **MMSE Approximation**
   - Previous Work

3. **Stochastic Resonance**
   - Can Noise Help Denoising?

4. **Our Proposed Method**
   - The Algorithm
   - Unitary Case Analysis
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5. Conclusions
Random OMP

A modified version of the OMP algorithm.

OMP: picks the atom most correlated with the current residual.

RandOMP: weights the unpicked atoms according to their correlation with the residual, and chooses randomly.

Repeated many times leading to a variety of solutions.

Averages the solutions to retrieve a final estimate.

Asymptotically converges with the MMSE estimator when:

- The dictionary is unitary.
- The cardinality of the sparse representation is 1.

Empirically achieves better MSE than OMP even when these conditions are not met.

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Why do these methods work?

$p(s|y)$ has an exponential nature. \[ \hat{\alpha}_{\text{MMSE}} = \sum_{s \in \{0, 1\}} m p(s|y) \hat{\alpha}_{\text{Oracle}} s \]

These methods find a "dominant" subset $\omega$ of supports and approximate their weights (posterior probabilities).

Previously suggested algorithms operate in a greedy fashion. \[ \Rightarrow \text{They are impractical for high dimensional signals}. \]
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MMSE Approximation Methods

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Stochastic Resonance

Definition

- Originally, was suggested as an explanation to the periodic recurrence of ice ages.
- Today, broadly applied to describe a more general phenomenon where presence of noise in a *nonlinear* system provides a better response.
### Definition

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Noise **improves** system performance?
Dither

- In signal quantization, additive noise is used to create stochastic quantization error.
- For example in images, dither prevents color banding which creates unpleasant images.
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Dither – Noise Has a Constructive Value

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The Proposed Algorithm

\[ y \]
The Proposed Algorithm

\[ y \xrightarrow{+} n \xrightarrow{+} y \]
The Proposed Algorithm

\[ y + n \rightarrow \text{MAP} \]
The Proposed Algorithm

\[ y + n \rightarrow \text{MAP} \rightarrow S \]
The Proposed Algorithm
The Proposed Algorithm

y + n
S
MAP
Oracle

y → + → MAP

Oracle → S
The Proposed Algorithm
Algorithm 1: Stochastic Resonance MMSE Approximation

\textbf{input} : \( y, D, \text{PursuitMethod}, \sigma_n, K \)

\begin{algorithmic}
\FOR {\( k \in 1 \ldots K \)}
    \STATE \( n_k \leftarrow \text{SampleNoise}(\sigma_n) \)
    \STATE \( \tilde{\alpha}_k \leftarrow \text{PursuitMethod}(y + n_k, D) \)
    \STATE \( \hat{S}_k \leftarrow \text{Support}(\tilde{\alpha}_k) \)
    \STATE \( \hat{\alpha}_k \leftarrow \hat{\alpha}_{\text{Oracle}}(\hat{S}_K(y)) \)
\ENDFOR
\STATE \( \hat{\alpha} \leftarrow \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k \)
\end{algorithmic}
The Proposed Algorithm

Algorithm 1: Stochastic Resonance MMSE Approximation

input : $y, D, \text{PursuitMethod}, \sigma_n, K$

output: $\hat{\alpha}$

for $k \in 1 \ldots K$
do

$n_k \leftarrow \text{SampleNoise}(\sigma_n)$

$\tilde{\alpha}_k \leftarrow \text{PursuitMethod}(y + n_k, D)$

$\hat{S}_k \leftarrow \text{Support}(\tilde{\alpha}_k)$

$\hat{\alpha}_k \leftarrow \hat{\alpha}_{\text{Oracle}} \hat{S}_K(y)$
endo
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Algorithm 1: Stochastic Resonance MMSE Approximation

**input**: $y, D, \text{PursuitMethod}, \sigma_n, K$

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**for** $k \in 1...K$ **do**

- $n_k \leftarrow \text{SampleNoise}(\sigma_n)$
- $\tilde{\alpha}_k \leftarrow \text{PursuitMethod}(y + n_k, D)$
- $\hat{S}_k \leftarrow \text{Support}(\tilde{\alpha}_k)$
- $\hat{\alpha}_k \leftarrow \hat{\alpha}_{\text{Oracle}}(y)$

$\hat{\alpha} \leftarrow \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k$
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**Algorithm 1: Stochastic Resonance MMSE Approximation**

**input**: $y, D, \text{PursuitMethod}, \sigma_n, K$

**output**: $\hat{\alpha}$

for $k \in 1...K$ do

\[ n_k \leftarrow \text{SampleNoise}(\sigma_n) \]

end
Algorithm 1: Stochastic Resonance MMSE Approximation

**input**: \( y, D, \text{PursuitMethod}, \sigma_n, K \)

**output**: \( \hat{\alpha} \)

**for** \( k \in 1...K \) **do**

\[
\begin{align*}
  n_k &\leftarrow \text{SampleNoise}(\sigma_n) \\
  \tilde{\alpha}_k &\leftarrow \text{PursuitMethod}(y + n_k, D)
\end{align*}
\]

\( \hat{\alpha} \leftarrow \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k \)
Algorithm 1: Stochastic Resonance MMSE Approximation

input : $y$, $D$, PursuitMethod, $\sigma_n$, $K$
output: $\hat{\alpha}$

for $k \in 1...K$ do

$n_k \leftarrow \text{SampleNoise}(\sigma_n)$
$
\tilde{\alpha}_k \leftarrow \text{PursuitMethod}(y + n_k, D)$

$\hat{S}_k \leftarrow \text{Support}(\tilde{\alpha}_k)$

end
The Proposed Algorithm

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**input**: \( y, D, \text{PursuitMethod}, \sigma_n, K \)

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for \( k \in 1 \ldots K \) do

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    \tilde{\alpha}_k & \leftarrow \text{PursuitMethod}(y + n_k, D) \\
    \hat{S}_k & \leftarrow \text{Support}(\tilde{\alpha}_k) \\
    \hat{\alpha}_k & \leftarrow \hat{\alpha}_{\hat{S}_k}^{\text{Oracle}}(y)
\end{align*}
\]

end
Algorithm 1: Stochastic Resonance MMSE Approximation

**input** : $y, D, \text{PursuitMethod}, \sigma_n, K$  
**output:** $\hat{\alpha}$

for $k \in 1...K$ do
  $n_k \leftarrow \text{SampleNoise}(\sigma_n)$  
  $\tilde{\alpha}_k \leftarrow \text{PursuitMethod}(y + n_k, D)$  
  $\hat{S}_k \leftarrow \text{Support}(\tilde{\alpha}_k)$  
  $\hat{\alpha}_k \leftarrow \hat{\alpha}^\text{Oracle}_{\hat{S}_K}(y)$
end

$\hat{\alpha} \leftarrow \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k$
The Proposed Algorithm – Does It Work?

- \( D \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
The Proposed Algorithm – Does It Work?

- \( D \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
- \( \|\alpha\|_0 = 1, \alpha_s \sim \mathcal{N}(0, 1) \).

\[ \nu \sim \mathcal{N}(0, \sigma^2 \nu I_{50}), \sigma^2 = 0.2. \]

0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40

\[ 2.0 \quad 2.5 \quad 3.0 \quad 3.5 \quad 4.0 \quad 4.5 \quad 5.0 \]

MSE

SR Estimators
MMSE

MSE for Sparse Prior

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The Proposed Algorithm – Does It Work?

- \( D \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
- \( \| \alpha \|_0 = 1, \alpha_s \sim \mathcal{N}(0, 1) \).
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The Proposed Algorithm – Does It Work?

- $D \in \mathbb{R}^{50 \times 100}$ a normalized random dictionary.
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- 100 iterations of stochastic resonance.
The Proposed Algorithm – Does It Work?

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- 100 iterations of stochastic resonance.

![Graph showing SR Estimators: MAP, SR, MMSE vs. MSE and \( \sigma_n \)]
1. Bayesian Framework
   - The Generative Model
   - Bayesian Estimators

2. MMSE Approximation
   - Previous Work

3. Stochastic Resonance
   - Can Noise Help Denoising?

4. Our Proposed Method
   - The Algorithm
   - Unitary Case Analysis
   - The General Dictionary Case
   - Image Denoising

5. Conclusions
When $D$ is a unitary matrix ($D^T D = I$), the oracle, MAP and MMSE estimators are element-wise shrinkage operators:

\[ c \text{ and } \lambda_{\text{MAP}} \text{ depend on } p_i, \sigma_\alpha \text{ and } \sigma_\nu. \]
When $D$ is a unitary matrix ($D^T D = I$), the oracle, MAP and MMSE estimators are element-wise shrinkage operators:

$$\hat{\alpha}_{s}^{\text{Oracle}}(y) = c^2 D_s^T y = c^2 \beta_s$$

$c$ and $\lambda_{\text{MAP}}$ depend on $p_i$, $\sigma_\alpha$ and $\sigma_\nu$. 
When $\mathbf{D}$ is a unitary matrix ($\mathbf{D}^T \mathbf{D} = \mathbf{I}$), the oracle, MAP and MMSE estimators are element-wise shrinkage operators:

$$\hat{\alpha}_{\text{MAP}} (\mathbf{y}) = \mathcal{H}_{\lambda_{\text{MAP}}} (\mathbf{D}^T \mathbf{y}) = \mathcal{H}_{\lambda_{\text{MAP}}} (\beta)$$

$c$ and $\lambda_{\text{MAP}}$ depend on $p_i, \sigma_\alpha$ and $\sigma_\nu$. 
When $D$ is a unitary matrix ($D^T D = I$), the oracle, MAP and MMSE estimators are element-wise shrinkage operators:

\[
\hat{\alpha}_{\text{MAP}}(y) = \mathcal{H}_{\lambda_{\text{MAP}}}(D^T y) = \mathcal{H}_{\lambda_{\text{MAP}}}(\beta)
\]

\[
\mathcal{H}_{\lambda_{\text{MAP}}}(\beta_i) = \begin{cases} 
  c^2 \beta_i & \text{if } |\beta_i| \geq \lambda_{\text{MAP}}, \\
  0 & \text{else}
\end{cases}
\]

$c$ and $\lambda_{\text{MAP}}$ depend on $p_i, \sigma_\alpha$ and $\sigma_\nu$. 

Dror Simon (Technion)

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Unitary Case - Estimators

When $\mathbf{D}$ is a unitary matrix ($\mathbf{D}^T \mathbf{D} = \mathbf{I}$), the oracle, MAP and MMSE estimators are element-wise shrinkage operators:

$$\hat{\alpha}_i^{\text{MMSE}}(\beta_i) = \frac{\exp \left( \frac{c^2}{2\sigma_\nu^2} \beta_i^2 \right) \frac{p_i}{1-p_i} \sqrt{1 - c^2}}{1 + \exp \left( \frac{c^2}{2\sigma_\nu^2} \beta_i^2 \right) \frac{p_i}{1-p_i} \sqrt{1 - c^2}} c^2 \beta_i$$

$c$ and $\lambda_{\text{MAP}}$ depend on $p_i$, $\sigma_\alpha$ and $\sigma_\nu$. 

Unitary Case – Estimators

MMSE and MAP shrinkage curves

\[ \hat{\alpha} \]

\[ \beta \]

MMSE

MAP
To use our proposed algorithm we need to provide a pursuit.
To use our proposed algorithm we need to provide a pursuit. The MAP is attainable.
To use our proposed algorithm we need to provide a pursuit. The MAP is attainable. \(\implies\) Use it!
To use our proposed algorithm we need to provide a pursuit. The MAP is attainable. ⇒ Use it!

### Subtractive hard-thresholding $\mathcal{H}^- (\cdot)$

$$\mathcal{H}^- (\beta, \tilde{n}) = \begin{cases} c_2 \beta & \text{if } |\beta + \tilde{n}| \geq \lambda_{MAP}, \\ 0 & \text{otherwise.} \end{cases}$$
To use our proposed algorithm we need to provide a pursuit. The MAP is attainable. ➞ Use it!

**Subtractive hard-thresholding** \( \mathcal{H}^- (\cdot) \)

\[
\mathcal{H}^- (\beta, \tilde{n}) = \begin{cases} 
c^2 \beta & \text{if } |\beta + \tilde{n}| \geq \lambda_{\text{MAP}}, \\
0 & \text{otherwise.}
\end{cases}
\]
To use our proposed algorithm we need to provide a pursuit. The MAP is attainable. \( \implies \) Use it!

**Subtractive hard-thresholding** \( H^- (\cdot) \)

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H^- (\beta, \tilde{n}) = \begin{cases} 
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\]
To use our proposed algorithm we need to provide a pursuit. The MAP is attainable. \[\Rightarrow\] Use it!

Subtractive hard-thresholding \(\mathcal{H}^- (\cdot)\)

\[
\mathcal{H}^- (\beta, \tilde{n}) = \begin{cases} 
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0 & \text{otherwise}.
\end{cases}
\]
What happens as $K \to \infty$?
What happens as $K \rightarrow \infty$?

$$\hat{\alpha}_k (\beta, \tilde{n}_k) = \mathcal{H}^- (\beta, \tilde{n}_k) = \begin{cases} 
c^2 \beta & \text{if } |\beta + \tilde{n}_k| \geq \lambda, \\
0 & \text{otherwise.}
\end{cases}$$
Unitary Case – Asymptotic Estimator

What happens as $K \to \infty$?

\[
\hat{\alpha}_k (\beta, \tilde{n}_k) = H^-(\beta, \tilde{n}_k) = \begin{cases} 
c^2 \beta & \text{if } |\beta + \tilde{n}_k| \geq \lambda, \\
0 & \text{otherwise.}
\end{cases}
\]

\[
\hat{\alpha} = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k = \mathbb{E}_n \{\hat{\alpha}_k\}
\]
What happens as $K \to \infty$?

$$\hat{\alpha}_k (\beta, \tilde{n}_k) = \mathcal{H}^{-}(\beta, \tilde{n}_k) = \begin{cases} c^2 \beta & \text{if } |\beta + \tilde{n}_k| \geq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{\alpha} = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k = \mathbb{E}_n \{ \hat{\alpha}_k \} = \int_{-\infty}^{\infty} \mathcal{H}^{-}(\beta, \tilde{n}) p(\tilde{n}) d\tilde{n}$$
What happens as $K \to \infty$?

$$\hat{\alpha}_k (\beta, \tilde{n}_k) = \mathcal{H}^{-} (\beta, \tilde{n}_k) = \begin{cases} c^2 \beta & \text{if } |\beta + \tilde{n}_k| \geq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

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$$= \ldots$$
What happens as $K \to \infty$?

\[ \hat{\alpha}_k (\beta, \tilde{n}_k) = H^- (\beta, \tilde{n}_k) = \begin{cases} c^2 \beta & \text{if } |\beta + \tilde{n}_k| \geq \lambda, \\ 0 & \text{otherwise}. \end{cases} \]

\[ \hat{\alpha} = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \hat{\alpha}_k = \mathbb{E}_n \{ \hat{\alpha}_k \} = \int_{-\infty}^{\infty} H^- (\beta, \tilde{n}) p (\tilde{n}) d\tilde{n} \]

\[ = \ldots \]

\[ = c^2 \beta \left[ Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta}{\sigma_n} \right) \right] \]

\[ Q(x) = \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt \]
How does this estimator perform?
Unitary Case – Empirical Performance

How does this estimator perform?

![MMSE vs Subtractive SR](image)
How does this estimator perform?

![Graph](image-url)
Stochastic Resonance vs. MMSE

Are the two the same?

Stochastic Resonance & MMSE

\[ \hat{\alpha}_{\text{stochastic}} = c^2 \beta \left[ Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta}{\sigma_n} \right) \right] \]

\[ \hat{\alpha}_{\text{MMSE}} = \exp \left( \frac{c^2}{2\sigma_v^2} \beta^2 \right) \frac{p_i}{1-p_i} \sqrt{1-c^2} \frac{c^2 \beta}{1 + \exp \left( \frac{c^2}{2\sigma_v^2} \beta^2 \right) \frac{p_i}{1-p_i} \sqrt{1-c^2}} \]

No... But are they close?
Empirically yes, but for the right choice of parameters.

We can set the parameters by using SURE.

More information in our paper.
Stochastic Resonance vs. MMSE

Are the two the same?

**Stochastic Resonance & MMSE**

\[
\hat{\alpha}_{\text{stochastic}} = c^2 \beta \left[ Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta}{\sigma_n} \right) \right]
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\[
\hat{\alpha}_{\text{MMSE}} = \frac{\exp \left( \frac{c^2}{2\sigma_v^2} \beta^2 \right) \frac{p_i}{1-p_i} \sqrt{1 - c^2}}{1 + \exp \left( \frac{c^2}{2\sigma_v^2} \beta^2 \right) \frac{p_i}{1-p_i} \sqrt{1 - c^2}} c^2 \beta
\]

No... But are they close?
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Are the two the same?

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\[ \hat{\alpha}_{\text{stochastic}} = c^2 \beta \left[ Q \left( \frac{\lambda + \beta}{\sigma_n} \right) + Q \left( \frac{\lambda - \beta}{\sigma_n} \right) \right] \]

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\[ \hat{\alpha}_{\text{MMSE}} = \frac{\exp \left( \frac{c^2 \beta^2}{2\eta I} \right) p_i}{1-p_i} \frac{\sqrt{1-c^2}}{\sqrt{1-c^2}} c^2 \beta \]

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Stochastic Resonance vs. MMSE

Are the two the same?

Stochastic Resonance & MMSE

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\hat{\alpha}_{\text{stochastic}} = c^2 \beta \left[ Q\left( \frac{\lambda + \beta}{\sigma_n} \right) + Q\left( \frac{\lambda - \beta}{\sigma_n} \right) \right]
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\hat{\alpha}_{\text{MMSE}} = \exp\left( \frac{c^2}{2\sigma_n^2} \beta^2 \right) \frac{p_i}{1-p_i} \sqrt{1 - c^2} \frac{1 + \exp\left( \frac{c^2}{2\sigma_n^2} \beta^2 \right) \frac{p_i}{1-p_i} \sqrt{1 - c^2}}{c^2 \beta}
\]

No... But are they close?
Empirically yes, but for the right choice of parameters.
We can set the parameters by using SURE.
More information in our paper.
Unitary Case – Summary

What about non-unitary cases?
What about non-unitary cases?
What about non-unitary cases?
The General Dictionary Case

The general case is harder to analyze. The MAP estimator is exhaustive ⇒ use a pursuit algorithm instead. The performance (and the analysis) depends on the pursuit used. We separate to two cases:

The generative model’s parameters are known.
The generative model’s parameters are not known.
The General Dictionary Case

- The general case is harder to analyze.

The MAP estimator is exhaustive \( \Rightarrow \) use a pursuit algorithm instead.
The performance (and the analysis) depends on the pursuit used.

We separate to two cases:

- The generative model's parameters are known.
- The generative model's parameters are not known.
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The General Dictionary Case

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The general case is harder to analyze.

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- We separate to two cases:
  - The generative model’s parameters are known.
The general case is harder to analyze.

The MAP estimator is exhaustive $\implies$ use a pursuit algorithm instead.

The performance (and the analysis) depends on the pursuit used.

We separate to two cases:

- The generative model’s parameters are known.
- The generative model’s parameters are not known.
General Case – Known Parameters

The parameters are known ⇒ use them.

Replace the arithmetic mean in the proposed algorithm with a weighted sum. The weights are obtained by:

- The occurrence of each support.
- The unnormalized posterior probability \( p(s) p(y|s) \).

In our paper we prove it is equivalent to a Monte Carlo importance sampling simulation. ⇒ Asymptotically converges to the MMSE!
The parameters are known \implies use them.

Replace the arithmetic mean in the proposed algorithm with a weighted sum. The weights are obtained by:

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The weights are obtained by:

- The occurrence of each support.
- The unnormalized posterior probability $p(s)p(y|s)$.

In our paper we prove it is equivalent to a Monte Carlo importance sampling simulation.

$\implies$ Asymptotically converges to the MMSE!
General Case – Known Parameters

- $D \in \mathbb{R}^{50\times100}$ a normalized random dictionary.
- $\|\alpha\|_0 = 1, \alpha_s \sim \mathcal{N}(0, 1)$.
- $\nu \sim \mathcal{N}(0, \sigma^2_\nu I_{50}), \sigma_\nu = 0.2$.
- 35 iterations of stochastic resonance.
**General Case – Known Parameters**

- \( D \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
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- 35 iterations of stochastic resonance.

![Graph showing MSE for SR Estimators](image)

**SR Estimators**

- MAP
- Subtractive SR
- Importance Sampling
- MMSE

**MSE** vs. **\( \sigma_n \)**
Parameters’ values are lacking.

How can we estimate a support?
Use any pursuit (MAP approximation).

How can we obtain the oracle estimator?
Simply use least squares

\[
\hat{\alpha}(s, y)_{\text{oracle}} = (D^T s D s)^{-1} D^T s y.
\]
Parameters’ values are lacking. \(\implies\) the same MMSE is no longer attainable.
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How can we estimate a support?
• Parameters’ values are lacking. \(\implies\) the same MMSE is no longer attainable.

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Parameters’ values are lacking. \( \implies \) the same MMSE is no longer attainable.

- How can we estimate a support? Use any pursuit (MAP approximation).
- How can we obtain the oracle estimator?
Parameters’ values are lacking. $\implies$ the same MMSE is no longer attainable.

How can we estimate a support? Use any pursuit (MAP approximation).

How can we obtain the oracle estimator? Simply use least squares

$$\hat{\alpha}(s, y)_{\text{oracle}} = (D_s^T D_s)^{-1} D_s^T y.$$
General Case – Unknown Parameters

Oracle

\[ y + n = S \]

\[ \text{MAP} \]

Dror Simon (Technion)
General Case – Unknown Parameters

\[ y + n \rightarrow \text{Pursuit Algo.} \rightarrow S \rightarrow \text{Least Squares} \rightarrow y + n \]

Dror Simon (Technion)
General Case – Unknown Parameters

- \( \mathbf{D} \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
- \( \| \mathbf{\alpha} \|_0 = 1, \mathbf{\alpha}_s \sim \mathcal{N}(0, 1). \)
- \( \nu \sim \mathcal{N}(\mathbf{0}, \sigma^2_{\nu} \mathbf{I}_{50}) , \sigma_{\nu} = 0.2. \)
- 100 iterations of stochastic resonance.
**General Case – Unknown Parameters**

- \( \mathbf{D} \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
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General Case – Unknown Parameters

- \( \mathbf{D} \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
- \( \| \mathbf{\alpha} \|_0 = 1 \), \( p_i = 0.05 \), \( \mathbf{\alpha}_s \sim \mathcal{N}(0, 1) \).
- \( \mathbf{\nu} \sim \mathcal{N}(\mathbf{0}, \sigma_{\nu}^2 \mathbf{I}_{50}) \), \( \sigma_{\nu} = 0.2 \).
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General Case – Unknown Parameters

- \( \mathbf{D} \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
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- \( \nu \sim \mathcal{N}(0, \sigma_{\nu}^2 \mathbf{I}_{50}) \), \( \sigma_{\nu} = 0.2 \).
- 100 iterations of stochastic resonance.

Use bounded noise formulation for the pursuit:

(Omp) \( \min_{\alpha} \|\alpha\|_0 \) s.t. \( \|\mathbf{y} - \mathbf{D}\alpha\|_2 \leq \epsilon \),

(Bp) \( \min_{\alpha} \|\alpha\|_1 \) s.t. \( \|\mathbf{y} - \mathbf{D}\alpha\|_2 \leq \epsilon \).
General Case – Unknown Parameters

- \( D \in \mathbb{R}^{50 \times 100} \) a normalized random dictionary.
- \( \|\alpha\|_0 = 1 \) \( p_i = 0.05 \), \( \alpha_s \sim \mathcal{N}(0, 1) \).
- \( \nu \sim \mathcal{N}(0, \sigma^2_{\nu} I_{50}) \), \( \sigma_\nu = 0.2 \).
- 100 iterations of stochastic resonance.
Outline

1. Bayesian Framework
   - The Generative Model
   - Bayesian Estimators

2. MMSE Approximation
   - Previous Work

3. Stochastic Resonance
   - Can Noise Help Denoising?

4. Our Proposed Method
   - The Algorithm
   - Unitary Case Analysis
   - The General Dictionary Case
   - Image Denoising

5. Conclusions
Method Used:

6Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Method Used:

- Dataset containing facial images.

---


6 Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Method Used:

- Dataset containing facial images.
- Trained a Trainlet\(^5\) dictionary on clean facial images.


\(^6\)Dai, Wei, and Olgica Milenkovic, 2009. ”Subspace pursuit for compressive sensing signal reconstruction.”
Method Used:

- Dataset containing facial images.
- Trained a Trainlet⁵ dictionary on clean facial images.
- Pursuit used: Subspace Pursuit (SP)⁶.

---

⁶Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Method Used:

- Dataset containing facial images.
- Trained a Trainlet\(^5\) dictionary on clean facial images.
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Experiment:

\(^6\)Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Method Used:

- Dataset containing facial images.
- Trained a Trainlet\textsuperscript{5} dictionary on clean facial images.
- Pursuit used: Subspace Pursuit (SP)\textsuperscript{6}.

Experiment:

- Added noise to an unseen image.

\textsuperscript{6}Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Image Denoising

Method Used:

- Dataset containing facial images.
- Trained a Trainlet\(^5\) dictionary on clean facial images.
- Pursuit used: Subspace Pursuit (SP)\(^6\).

Experiment:

- Added noise to an unseen image.
- Use the dictionary and SP to denoise. Tune L parameter to obtain optimal denoising performance.

---


\(^6\)Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Method Used:

- Dataset containing facial images.
- Trained a Trainlet\(^5\) dictionary on clean facial images.
- Pursuit used: Subspace Pursuit (SP)\(^6\).

Experiment:

- Added noise to an unseen image.
- Use the dictionary and SP to denoise. Tune L parameter to obtain optimal denoising performance.
- Use SR algorithm using the same SP configuration used in the previous step.

---


\(^6\)Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."
Image Denoising – Results

Noisy image. PSNR=16.1 dB.
Subspace Pursuit. PSNR=26.88 dB.
Stochastic Resonance. PSNR=28.76 dB.

∼2dB better.

Clean Image.
Noisy image.
PSNR=16.1 dB.

Clean Image.
∼2dB better.
Image Denoising – Results

Noisy image. PSNR=16.1 dB.

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5 Conclusions
Conclusions

The MMSE estimator is desired, but involves an exhaustive computation. MAP estimation is hard as well, but many approximation methods are available. We can use synthetic noise and any MAP estimator approximation to achieve an MMSE estimator approximation. MMSE estimator approximation is attainable, even for large dimensions.
Conclusions

- MMSE estimator is desired, but involves an exhaustive computation.
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- MAP is hard as well, but many approximation methods are available.
- We can use synthetic noise and any MAP estimator approximation to achieve an MMSE estimator approximation.
- MMSE estimator approximation is attainable, even for large dimensions.
Thank You